Alternate Current Measurement.

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The lack of precision of measurements with alternate currents, as compared with those using direct currents, is mainly due to the relative sensitiveness of the instruments available for such tests. The fact that the turning moment acting on the moving system depends in one case on the square of the current and in the other on the first power of the current, readily explains the high ratio between the currents needed to cause the minimum measurable deflection in the two cases, but this ratio is, nevertheless, most striking when a numerical comparison is actually made on some fair basis. The only likely way at present of improving alternate current instruments is to use iron cored electromagnets to increase the strength of the magnetic field. I have found that the difficulties due to varying permeability and hysteresis of the iron can be avoided by exciting the electromagnet in shunt. It proves possible, with careful design, to construct an electromagnet whose flux is connected with the exciting voltage by a strict mathematical law involving no variable physical properties like permeability, etc. Such an electromagnet is eminently suited for measuring purposes. The theoretical and experimental study of instruments constructed on this principle has brought out certain novel points which are set forth in the present paper.

The first part discusses the mathematical relations of cyclic quantities having a common fundamental period, and constitutes a development of a method already published.* This method is the only one known to me which is independent of assumptions in regard to the wave form of the quantities dealt with. The usual methods, which are based on the erroneous assumptions of sine law wave form, are not any simpler in working, and are most unsatisfactory when the accuracy of new results has to be critically examined. All alternate current measurements refer to mean squares or to mean products, and the natural method of obtaining the connections between

* 'Roy. Soc. Proc.,' vol. 61, 1897.

such squares and products is to study the properties of quadratic functions of the variables. The earliest instance of this in alternate current theory was in connection with the "three voltmeter method."* Such processes lead to a very simple form of calculus appropriate to cyclic quantities.

The theory of the action of an alternating magnetic field on a movable coil conveying a current of the same frequency is discussed in the second part of the paper; and the application of such theory to shunt magnet instruments is given in the third part. The final portion deals with experimental verification.

Part I.—The Mathematical Relationships of Cyclic Quantities.

We shall use heavy letters such as $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, to denote single valued cyclic functions of the variable t, and the corresponding letters in lighter type, A, B, C, to denote the square root of mean square values of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, respectively. A we define as the *magnitude* of $\mathbf{A}$, and will be a constant as regards t. The mean value of the product of two quantities such as $\mathbf{A}$, $\mathbf{B}$, will be represented by $\overline{\mathbf{AB}}$, so that $\mathrm{A}^2 = \overline{\mathbf{A} \cdot \mathbf{A}} = \overline{\mathbf{A}^2}$.

Any cyclic quantity $\mathbf{A}$ must either have its mean value zero, when it may be described as *alternating*, or it must consist of the sum of two parts, one of which is *steady* and the other *alternating*. With few exceptions the quantities occurring in actual alternate current problems are alternating in the above sense. The differential coefficient of any cyclic quantity is necessarily alternating. The integral of a cyclic quantity is not met with unless its mean or steady value is zero, and as this integral will in most actual cases also have zero mean value it will be as definite a quantity as that of a differential coefficient. The product of two cyclic quantities such as a voltage $\mathbf{V}$ and a current $\mathbf{A}$ consists in general of two portions, one of which is steady and the other alternating. These parts correspond precisely with the scalar and vector products of two vectors.

Whatever cyclic quantities $\mathbf{A}$ and $\mathbf{B}$ may be if we consider the mean values of the two identical expressions

$$\left(\frac{\mathbf{A}}{\mathrm{A}}-\frac{\mathbf{B}}{\mathrm{B}}\right)^2 \equiv \frac{\mathbf{A}^2}{\mathrm{A}^2}+\frac{\mathbf{B}^2}{\mathrm{B}^2}-2\frac{\mathbf{AB}}{\mathrm{AB}},$$

we see that these mean values can only be zero when $\mathbf{A}$ is proportional to $\mathbf{B}$ for *each instant* of the cycle, and also that we can always find a *real* angle ϕ such that

$$\overline{\mathbf{AB}} = \mathrm{AB}\cos\phi. \tag{1}$$

The value of ϕ determined from this equation may be defined as the *phase*

* 'Roy. Soc. Proc.,' vol. 49, March, 1891.

difference of the quantities $\mathbf{A}$ and $\mathbf{B}$. These quantities will be said to be *in phase* when ϕ is zero, and this can only happen when the ratio of $\mathbf{A}$ to $\mathbf{B}$ is independent of the time t. They will be said to be *in quadrature* when $\cos\phi$ is zero, that is, when the mean product of $\mathbf{A}$ and $\mathbf{B}$ is zero.

For any two quantities $\mathbf{A}$ and $\mathbf{N}$ it is always possible to consider either of them, say $\mathbf{N}$, as the sum of two quantities the first of which is in phase with $\mathbf{A}$ and the second in quadrature with it, for we can so define $\mathbf{n}$ that

(2) $$\mathbf{N} = \mathrm{L}\mathbf{A}+\mathbf{n},$$

where L is a quantity, independent of time, which we can so choose that $\mathbf{n}$ and $\mathbf{A}$ are in quadrature. It readily follows that the magnitudes of the three quantities in (2) can be denoted by the lengths of the three sides of a right-angled triangle, and that the angle between any two sides of this triangle is the phase difference of the corresponding quantities. The mode of proof merely involves the process of multiplying the equation by some cyclic quantity and taking means. By successive application of this process, it is possible to establish the following theorem:—

Each one of a number of known cyclic quantities, however different these may be in wave form, can be expressed as a linear function of an equal number of other cyclic quantities, these latter being such that the mean square of each is unity and the mean product of any two is zero.

Thus, if there are n cyclic quantities $\mathbf{A}_1$, $\mathbf{A}_2, \ldots \mathbf{A}_n$, we can always find n other cyclic quanties, $\mathbf{x}_1$, $\mathbf{x}_2, \ldots \mathbf{x}_n$, each of which is of unit magnitude, each two of which are in quadrature, and which are such as to satisfy the identities

(3) $$\begin{aligned} \mathbf{A}_1 &\equiv \mathrm{A}_1\mathbf{x}_1 \\ \mathbf{A}_2 &\equiv \mathrm{A}_2[\mathbf{x}_1\cos\alpha_{21}+\mathbf{x}_2\cos\alpha_{22}] \\ &\cdots\cdots\cdots\cdots \\ \mathbf{A}_n &\equiv \mathrm{A}_n[\mathbf{x}_1\cos\alpha_{n1}+\mathbf{x}_2\cos\alpha_{n2}+\ldots+\mathbf{x}_n\cos\alpha_{nn}], \end{aligned}$$

where A_m is the magnitude of $\mathbf{A}_m$, and where

(4) $$1 = \cos^2\alpha_{m1}+\cos^2\alpha_{m2}+\ldots+\cos^2\alpha_{mm}$$

for every value of m from 1 to n.

The quantities α are perfectly determined by the magnitudes and phase differences of the quantities $\mathbf{A}$. Indeed, α_{rs} is the phase difference between $\mathbf{A}_r$ and $\mathbf{x}_s$, so that if the system of equations (3) has been established as far as, say, $\mathbf{A}_3$, the quantities $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ are known; the phase differences between these and $\mathbf{A}_4$ can be calculated, determining α_{41}, α_{42}, α_{43}; the equation for $\mathbf{A}_4$, constituting the definition of $\mathbf{x}_4$, can then be used to show that $\mathbf{x}_4$ is in quadrature with each of the quantities $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$; equation (4) for $m = 4$ determines α_{44}; and so on.

If three quantities, $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, of different wave form, together with linear functions of these quantities, comprise all the cyclic functions which have to be considered, it is possible, as previously shown,* to represent the magnitudes and phase relationships of all such quantities by a vector figure drawn in three dimensions. This is not possible when four independent wave forms are involved, but in all cases it is possible to establish the system of equations (3) and (4).

Important alternate current problems involve so many quantities, and the relationship between these is so complicated by the effects of varying permeability, hysteresis, irregular wave forms, etc., that mathematically accurate representation is impossible with a simpler system of equations than is indicated in (3). In practice, however, great simplification results from two considerations. In the first place, although the quantities involved may be very numerous, it is only necessary, as a rule, to consider the mutual relations of two or three of them at any one time, and it is in general possible to construct the vector figure actually needed for this purpose. In the second place, the figure can in all cases be reduced to a two-dimensioned figure by projection, as illustrated by figs. 1, 2, and 3 below. Each quantity involved can be reduced to one of the forms

$$\mathbf{R} = \mathrm{R}\,[\mathbf{x}\cos\phi+\mathbf{y}\sin\phi],$$
$$\mathbf{S} = \mathrm{S}\,[\mathbf{x}\cos\alpha+\mathbf{y}\cos\beta+\mathbf{z}\cos\gamma],$$

where $\mathbf{x}$ and $\mathbf{y}$ are cyclic quantities in quadrature, the same for all the quantities $\mathbf{R}$, $\mathbf{S}$, and where each quantity $\mathbf{z}$ is in quadrature both with $\mathbf{x}$ and $\mathbf{y}$. The projection of $\mathbf{S}$, denoted by $\mathbf{S}_1$, is

$$\mathbf{S}_1 = \mathrm{S}\,[\mathbf{x}\cos\alpha+\mathbf{y}\cos\beta].$$

It is possible to represent all the quantities $\mathbf{R}$, $\mathbf{S}_1$, by vectors in a plane, and it is always true that

(5) $$\overline{\mathbf{RS}} = \overline{\mathbf{RS}_1}.$$

The plane figure so constructed will in most cases meet all requirements.

It is frequently necessary to consider quantities like $\mathbf{N}$ and $\dot{\mathbf{N}}$, such that one is the differential coefficient of the other. Newton's notation may be conveniently used, not only for the instantaneous values $\mathbf{N}$, $\dot{\mathbf{N}}$, but also for the corresponding magnitudes N, $\dot{\mathrm{N}}$, since the latter are constants as regards time, so that no ambiguity can arise.

If $\mathbf{A}$ and $\mathbf{B}$ are any two cyclic quantities, it will be readily seen that

(6) $$\overline{\mathbf{A}\dot{\mathbf{B}}} = -\overline{\mathbf{B}\dot{\mathbf{A}}}, \qquad \overline{\mathbf{A}\dot{\mathbf{A}}} = 0.$$

* 'Roy. Soc. Proc.,' vol. 61, p. 465.

It follows from (2), that for any two quantities $\mathbf{N}$ and $\mathbf{A}$ we can so choose L that a right-angled vector triangle is denoted by the equation

$$\text{(i)} \quad \mathbf{N} = \mathrm{L}\mathbf{A} + \mathbf{n}.$$

From this we have

$$\text{(ii)} \quad \dot{\mathbf{N}} = \mathrm{L}\dot{\mathbf{A}} + \dot{\mathbf{n}}$$

and

$$\text{(iii)} \quad \dot{\mathbf{N}}_1 = \mathrm{L}\dot{\mathbf{A}}_1 + \dot{\mathbf{n}}_1.$$

The triangle represented by (iii) is the projection of that denoted by (ii) on the plane determined by (i) or by $\mathbf{N}$, $\mathbf{A}$, and $\mathbf{n}$. By making use of (5), it can be seen that the plane vectors involved in the above equations are

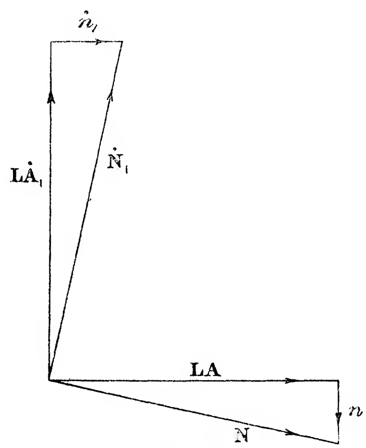

FIG. 1.

represented, as in fig. 1, by two similar right-angled triangles with corresponding sides perpendicular.

It follows that—

(7) If $\mathbf{N}$ and $\mathbf{A}$ are *any* two cyclic quantities represented by vectors in a plane, the vector projections of $\dot{\mathbf{N}}$ and $\dot{\mathbf{A}}$ on the same plane will be obtained by turning each of the vectors $\mathbf{N}$ and $\mathbf{A}$ through a right angle *in the same sense*, and by increasing their magnitudes *in the same proportion.*

We might similarly represent $\dot{\mathbf{N}}$, $\dot{\mathbf{A}}$, and $\dot{\mathbf{n}}$ by vectors in a plane, and project on to this plane vectors denoting $\mathbf{N}$, $\mathbf{A}$, and $\mathbf{n}$, and also those denoting $\ddot{\mathbf{N}}$, $\ddot{\mathbf{A}}$, and $\ddot{\mathbf{n}}$. Representing the projected vectors by the suffix unity, we have fig. 2, consisting of three similar right-angled triangles. $\ddot{\mathbf{N}}_1$ will be represented by a vector in the same direction as, but drawn in the opposite sense to, that which represents $\mathbf{N}_1$. This will also be true of the vectors $\ddot{\mathbf{A}}_1$ and $\mathbf{A}_1$, etc. Of course, the actual magnitude of $\dot{\mathbf{N}}$ will bear to that of $\mathbf{N}$ a ratio which is dependent on the wave form of $\mathbf{N}$, and this ratio will not necessarily be the same as that of the magnitudes of $\dot{\mathbf{A}}$ and $\mathbf{A}$, two other

quantities related in the same way. Indeed, this ratio is not the same for $\ddot{\mathbf{N}}$ and $\dot{\mathbf{N}}$ as it is for $\dot{\mathbf{N}}$ and $\mathbf{N}$, except in the special case in which each follows the simple sine law denoted by $\ddot{\mathbf{N}} = -p^2\mathbf{N}$.

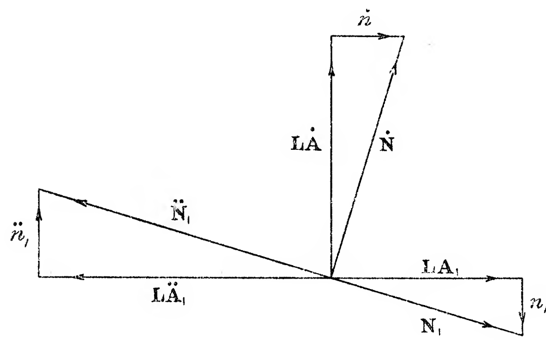

FIG. 2.

In other cases it can easily be established by aid of the theorems denoted by (1) and (6) that

$$
(8) \qquad \begin{cases} \mathrm{N} : \dot{\mathrm{N}} : \ddot{\mathrm{N}} : \dddot{\mathrm{N}}, \text{ etc.,} \\ = 1 : p_1 : p_1p_2 : p_1p_2p_3, \text{ etc.} \\ \text{where } p_1 < p_2 < p_3, \text{ etc.} \end{cases}
$$

It is also easy to show that

$$
(9) \qquad \ddot{\mathbf{N}} = -p_1{}^2\mathbf{N} + \mathbf{Z} \quad \text{and} \quad \ddot{\mathbf{N}}_1 = -p_1{}^2\mathbf{N},
$$

where $\mathbf{Z}$ is in quadrature with both $\mathbf{N}$ and $\dot{\mathbf{N}}$, and where $\ddot{\mathbf{N}}_1$ is the projection of $\ddot{\mathbf{N}}$ on the plane containing $\mathbf{N}$ and $\dot{\mathbf{N}}$.

Part II.—The Action of a Cyclic Magnetic Field on a Movable Coil conveying a Current of the same Frequency.

If $\mathbf{C}$ be the current traversing a coil, the flux through which is $\mathbf{F}$ (the flux per turn multiplied by the number of turns), and if θ be an angular displacement, the corresponding torque $\mathbf{T}$ will be given by

$$
(10) \qquad \mathbf{T} = \mathbf{C}\frac{d\mathbf{F}}{d\theta}.
$$

If the current $\mathbf{C}$ and the flux $\mathbf{F}$ are steady, the torque due to their interaction (by the convenient rule due to Maxwell, vol. II, § 489) will be so directed as to tend to increase $\mathbf{F}$, provided we consider $\mathbf{F}$ as positive when threading the coil in the same direction as the flux due to the current $\mathbf{C}$. If we choose a particular direction round the coil as positive for $\mathbf{C}$, this fixes the positive direction of $\mathbf{F}$, and the direction of $\mathbf{T}$, which is to be con-

sidered positive, is that corresponding with such a displacement $d\theta$ that $\mathbf{F}$ increases with the displacement. If the current $\mathbf{C}$ and the flux $\mathbf{F}$ are alternating, the above statements are still true at any instant, but it does not follow that the average or steady value of $\mathbf{T}$ is such as to turn the coil so as to increase the *magnitude* of $\mathbf{F}$, that is to say, so as to increase the square root of the time average of $\mathbf{F}^2$. The mean value of the product determining $\mathbf{T}$ will be zero when $\mathbf{C}$ and $\mathbf{F}$ are in quadrature, and will change sign as the phase difference alters from just below, to just above, a right angle.

If $\mathbf{F}$ is entirely due to $\mathbf{C}$, that is, if $\mathbf{F} = \mathrm{L}\mathbf{C}$, where L is the self-inductance of the coil, we have

$$\mathbf{T} = \mathbf{C}^2 \frac{d\mathrm{L}}{d\theta}, \tag{11}$$

and is necessarily positive, that is, $\mathbf{T}$ is so directed that the displacement it tends to produce is such as to increase L. This torque always exists whatever may be the cause of the current $\mathbf{C}$, but, in the cases we shall have to consider, it is so small as to be negligible.

We shall in all cases assume that the magnetic field, though varying with the time, has a fixed mode of distribution in space, or that the induction density at any point is the product of a vector function determined by the position of the point and not dependent on the time, and a scalar function of the time, the same for all points in the field. In other words, we shall assume that the fluxes through any two coils placed anywhere in the field are always in the same *phase*, though the *magnitude* of each flux depends on the configuration of the coil, and on the co-ordinates determining its position. Alternate currents of commercial frequencies vary so slowly that this assumption is justifiable, except in a few cases in which the variable permeability and hysteresis of iron prevent the medium from having a fixed magnetic character, and cause the flux distribution to alter for different magnetising currents.

If the position of the coil is completely determined by θ, we have, on the above assumptions,

$$\mathbf{F} = \mathrm{F}\tau,$$

where F depends solely on θ and τ depends solely on the time. Thus

$$\mathbf{T} = \mathbf{C}\frac{d\mathbf{F}}{d\theta} = \mathbf{C}\tau\frac{d\mathrm{F}}{d\theta},$$

or

$$\mathbf{T} = \frac{1}{\mathrm{F}}\frac{d\mathrm{F}}{d\theta}\mathbf{CF}. \tag{12}$$

Thus the mean value, or the steady part, of $\mathbf{T}$, which we shall denote by T_s, is given by

(13) $$T_s = \frac{1}{F}\frac{dF}{d\theta}\overline{\mathbf{CF}};$$

and this equation must be true in all cases.

Now if the current is $\mathbf{C}_1$, due to the electromotive force induced by the time rate of change of the moving coil flux $\mathbf{F}$, the torque which results will depend on the nature of the circuit containing this coil.

Suppose this circuit to be metallically closed, and to have a resistance R and self-inductance L, we then have

(14) $$R\mathbf{C}_1 + L\dot{\mathbf{C}}_1 = -\dot{\mathbf{F}}.$$

Multiply this by $\mathbf{F}$ and take means. We get

$$R\overline{\mathbf{C}_1\mathbf{F}} + L\overline{\dot{\mathbf{C}}_1\mathbf{F}} = -\overline{\mathbf{F}\dot{\mathbf{F}}} = 0;$$

but if we multiply the same equation by $L\mathbf{C}_1$ and take means, we get

$$LR\overline{\mathbf{C}_1^2} + L^2\overline{\mathbf{C}_1\dot{\mathbf{C}}_1} = -L\overline{\mathbf{C}_1\dot{\mathbf{F}}},$$

or, using (6), $$LRC_1^2 + 0 = +L\overline{\dot{\mathbf{C}}_1\mathbf{F}}.$$

By adding these two derived equations, we get, on dividing by R,

$$LC_1^2 + \overline{\mathbf{C}_1\mathbf{F}} = 0.$$

Hence, substituting in (13), we get

(15) $$T_s = -\frac{1}{F}\frac{dF}{d\theta}LC_1^2.$$

From this it follows that T_s is negative, or

(16) *A closed conducting circuit having self-inductance and placed in an alternating magnetic field will tend to set itself so as to decrease to a minimum the magnitude of the flux it surrounds.*

The forces tending to displace the coil vanish if L is zero or negligible, but they are not simply proportional to L, since $\mathbf{C}_1$ is inversely proportional to the impedance of the coil.

Suppose in the next place that the circuit of the coil is closed through a condenser of capacity K. If the resistance of the circuit is R, the current $\mathbf{C}_1$ induced by the field will, under ordinary circumstances, be so small that $R\mathbf{C}_1$ is negligible in comparison with $\dot{\mathbf{F}}$. Under these conditions, if $\mathbf{V}_1$ is the voltage of the condenser we find, after paying due regard to sign, that,

$$\mathbf{C}_1 = K\dot{\mathbf{V}}_1 \quad \text{and} \quad \mathbf{V}_1 = -\dot{\mathbf{F}},$$

whence $$\overline{\mathbf{C}_1\mathbf{F}} = K\overline{\dot{\mathbf{V}}_1\mathbf{F}} = -K\overline{\mathbf{V}_1\dot{\mathbf{F}}} = +K\overline{\mathbf{V}_1^2} = KV_1^2;$$

so that, using (13), we have in this case

(17) $$\mathrm{T}_s = +\frac{1}{\mathrm{F}}\frac{d\mathrm{F}}{d\theta}\mathrm{K}\mathrm{V}_1^2,$$

where V_1 is the voltage induced in the circuit by the field; thus T_s is necessarily positive, or we have

(18) *If a coil whose circuit is closed through a condenser be placed in an alternating magnetic field, it will tend to move so as to increase to a maximum the magnitude of the flux it surrounds.*

It can be shown that (17) is accurate, even allowing for the resistance of the coil, though in this case the voltage $\mathbf{V}_1$ of the condenser is not the same as $-\dot{\mathbf{F}}$ the electromotive force induced by the field. Allowing for the resistance R, it will be found that

$$\mathbf{C}_1 = \mathrm{K}\dot{\mathbf{V}}_1, \qquad \mathbf{V}_1 + \mathrm{R}\mathbf{C}_1 = -\dot{\mathbf{F}}.$$

From these equations we obtain

$$\overline{\mathbf{C}_1\mathbf{V}_1} = \mathrm{K}\overline{\mathbf{V}_1\dot{\mathbf{V}}_1} = 0,$$

$$\overline{\mathbf{C}_1\mathbf{F}} = \mathrm{K}\overline{\dot{\mathbf{V}}_1\mathbf{F}} = -\mathrm{K}\overline{\mathbf{V}_1\dot{\mathbf{F}}} = +\mathrm{K}\overline{\mathbf{V}_1(\mathbf{V}_1+\mathrm{R}\mathbf{C}_1)}$$
$$= \mathrm{K}\overline{\mathbf{V}}_1^2 = \mathrm{K}\mathrm{V}_1^2,$$

as in the previous case, while the *power* supplied to the circuit is

$$\overline{\mathbf{C}_1(-\dot{\mathbf{F}})} = \overline{\mathbf{C}_1(\mathbf{V}_1+\mathrm{R}\mathbf{C}_1)} = \mathrm{R}\mathrm{C}_1^2$$

and $$\dot{\mathrm{F}}^2 = \mathrm{V}_1^2 + \mathrm{R}^2\mathrm{C}_1^2,$$

so that if $\mathrm{R}\mathrm{C}_1$ is less than one per cent. of $\dot{\mathrm{F}}$, the values of V_1^2 and $\dot{\mathrm{F}}^2$ will differ by less than 1 part in 10,000.

Now let us suppose we have constructed an instrument for alternate current measurements, consisting of a movable coil placed in the intense alternating magnetic field due to an iron-cored electromagnet, and let us assume that the circuit of the coil includes a portion, external to the instrument, on which an alternating electromotive force $\mathbf{E}$ is impressed. The torque acting on the coil will still be given by (13), where $\mathbf{F}$ is the flux through the coil due to the electromagnet, and where $\mathbf{C}$ is the resultant current through the coil, due to all the impressed and reactive electromotive forces in the circuit.

Let us assume in the first place that the moving coil circuit is metallically closed, and that its resistance is R and its self-inductance L. We then have for the moving coil current

(19) $$\mathrm{R}\mathbf{C} + \mathrm{L}\dot{\mathbf{C}} = -\dot{\mathbf{F}} + \mathbf{E}.$$

The solution of this is $\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2$,

where $$\mathrm{R}\mathbf{C}_1 + \mathrm{L}\dot{\mathbf{C}}_1 = -\dot{\mathbf{F}}, \qquad \mathrm{R}\mathbf{C}_2 + \mathrm{L}\dot{\mathbf{C}}_2 = \mathbf{E},$$

so that the torque is

$$\mathrm{T}_s = \frac{1}{\mathrm{F}}\frac{d\mathrm{F}}{d\theta}\overline{\mathbf{CF}} = \frac{1}{\mathrm{F}}\frac{d\mathrm{F}}{d\theta}(\overline{\mathbf{C}_1\mathbf{F}} + \overline{\mathbf{C}_2\mathbf{F}}),$$

or

$$\mathrm{T}_s = \mathrm{T}_1 + \mathrm{T}_2,$$

where T_1 is the torque already calculated (15) which depends on $\mathbf{F}$, and is independent of $\mathbf{E}$, while T_2 is a torque due to the interaction of $\mathbf{F}$ with $\mathbf{C}_2$, a current calculated from $\mathbf{E}$ alone, without reference to $\mathbf{F}$, or to its reactive influence on the circuit.

A similar argument applies to the case in which the moving coil is closed through a condenser. In this case we have the equations

(20) $$\mathbf{C} = \mathrm{K}\dot{\mathbf{V}} \quad \text{and} \quad \mathbf{V} + \mathrm{R}\mathbf{C} + \mathrm{L}\dot{\mathbf{C}} = -\dot{\mathbf{F}} + \mathbf{E},$$

and we can put

$$\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2 \quad \text{and} \quad \mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2, \text{ etc.},$$

where the quantities having the first suffix satisfy the equations when $\mathbf{E}$ is zero, and those with the second suffix satisfy the equations when $\dot{\mathbf{F}}$ is put equal to zero. As in the previous case, the torque T_s on the moving coil will consist of two independent parts, one of which T_1 is solely due to $\mathbf{F}$, and the other of which T_2 is due to $\mathbf{C}_2$ and $\mathbf{F}$, where $\mathbf{C}_2$ is solely due to $\mathbf{E}$. In each case the importance or otherwise of T_1 can easily be tested in an actual instrument by simply arranging for $\mathbf{E}$ to be zero. As shown later, T_1 can also be easily calculated from the formula.

The method of superposition can be conveniently applied to alternate current problems provided, (1) the differential equations connecting the currents and voltages are linear, and (2) the resistances, induction coefficients, capacities and other factors are independent, not only of the time, but also of the physical variables, such as current, voltage, etc. Under these conditions it is easy to prove the following theorem:—

(21) If $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$, etc., be *any* impressed electromotive forces distributed in *any* manner among the branches of *any* network of conductors, the current $\mathbf{C}$ in *any* selected branch will be given *at each instant* by the equation

$$\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2 + \mathbf{C}_3 + \dots,$$

where $\mathbf{C}_1$ is the current calculated on the assumption that all the impressed electromotive forces are zero except $\mathbf{E}_1$, and similarly for $\mathbf{C}_2$, $\mathbf{C}_3$, etc. The current in any branch is at every instant the sum of the currents which would be produced in that branch by each impressed electromotive force acting alone.

Since the current in any branch can theoretically always be reduced to zero by inserting a suitable electromotive force into the branch, the above theorem may readily be used to prove that.

(22) If the electromotive forces impressed on a network cause a difference of potential $\mathbf{V}$ between two given points of the network, and if these two points be afterwards joined by a wire possessing resistance and self (but not mutual) inductance, the current which will flow through the wire will be exactly what an electromotive force $\mathbf{V}$ would produce in a circuit consisting of the wire, and the network to which the wire is joined, assuming the other impressed electromotive forces removed from the network.

The above theorems I have found serviceable in reasoning out the behaviour of conducting networks used for a number of tests made in connection with the present paper. The complete mathematical solution of the problems presented by such networks is, as a rule, impossible, owing to the unknown wave forms of the variables. For the purpose of an actual test the complete solution is not, as a rule, needed, while the particular relationship required can often be readily seen with the aid of the above theorems* without making any assumptions about wave form.

Part III.—The Theory of Shunt Magnet Instruments.

Let r be the resistance of the coil of an electromagnet subjected to a periodic voltage $\mathbf{V}$, and let $\mathbf{N}$ be the total number of lines of force enclosed by the winding, that is let $\mathbf{N}$ be the product of the number of turns of the coil and the flux of lines through the core. Let $\mathbf{A}_m$ be the current flowing through the coil. We then have

$$\mathbf{V} = r\mathbf{A}_m + \dot{\mathbf{N}}. \tag{23}$$

For the cores of large transformers with closed magnetic circuits it is in some cases the fact that the magnitude of $\mathbf{V}$ is over one hundred thousand times as great as that of $r\mathbf{A}_m$. For small magnetic circuits suitable for instruments of ordinary size the ratio will be much less, especially if an air gap is introduced to allow a coil to move across the field, but even in such cases I have found it possible with careful design to make this ratio exceed 250 for alternate currents having a frequency of 50 cycles per second.

Let us for the present assume r to be so small that rA_m is negligible compared with V, or that the ratio of the resistance to the impedance of the coil is negligible. The relationship between the field $\mathbf{N}$ and the exciting voltage $\mathbf{V}$ is then *independent of the permeability and hysteresis of the core.* Suppose the electromagnet to have a narrow air gap in which a coil can move,

* The truth of (21) and (22) for *direct current circuits* has been long known. The latter theorem (22) is due to Thévenin, 'Comptes Rendus,' 1888, vol. 97, p. 159, and is often convenient. Theorem (21) for *alternate current circuits*, though not precisely stated, seems more or less indicated in one of Heaviside's 'Electrical Papers,' vol. 2, pp. 294—296.

so as to turn about an axis, its position being completely specified by a deflection θ. Let $\mathbf{F}$ be the flux enclosed by this coil for the deflection θ. We assume $\mathbf{F}$ strictly in phase with $\mathbf{N}$, so that

(24) $$\mathbf{F} = \rho\mathbf{N},$$

where ρ depends on θ, but is independent of time. Hence, assuming r in (24) negligible, we have

(25) $$\dot{\mathbf{F}} = \rho\dot{\mathbf{N}} = \rho\mathbf{V},$$

so that ρ is the ratio of the voltage of the moving coil (on open circuit) to the applied voltage V, and is a quantity which for any deflection θ can be measured, provided suitable voltmeters are available.

Suppose the moving coil circuit be closed through a condenser of capacity K, and to have an electromotive force $\mathbf{E}$ impressed upon it, or suppose the moving coil (in series with the condenser K) be applied to mains at potential $\mathbf{E}$, we then have, from (20), assuming R and L negligible,

$$\mathbf{C} = \mathrm{K}\dot{\mathbf{V}}_1, \quad \text{where} \quad \mathbf{V}_1 = -\dot{\mathbf{F}}+\mathbf{E} = -\rho\mathbf{V}+\mathbf{E},$$

so that $$\mathbf{C} = -\mathrm{K}\rho\dot{\mathbf{V}}+\mathrm{K}\dot{\mathbf{E}}.$$

Hence $$\overline{\mathbf{FC}} = -\mathrm{K}\rho\overline{\dot{\mathbf{V}}\mathbf{F}}+\mathrm{K}\overline{\dot{\mathbf{E}}\mathbf{F}}$$

$$= +\mathrm{K}\rho\overline{\mathbf{V}\dot{\mathbf{F}}}-\mathrm{K}\overline{\mathbf{E}\dot{\mathbf{F}}};$$

and by (25) this is equal to

$$\overline{\mathbf{FC}} = \mathrm{K}\rho\,[\rho\mathbf{V}^2-\overline{\mathbf{EV}}].$$

For a voltmeter we can make $\mathbf{E}$ the same as $\mathbf{V}$, but we can apply it to the moving coil in either of two ways. Thus for a voltmeter

(26) $$\overline{\mathbf{FC}} = \mathrm{K}\rho\,(\rho\pm 1)\,\mathrm{V}^2.$$

This quantity is proportional to the torque, and hence this torque for a given value of θ and of ρ simply depends on V^2, so that the instrument can be calibrated as a voltmeter. It is easily possible to make ρ negligible compared with unity, but in any case the factor $[\rho\pm 1]$ only affects the calibration, and not the accuracy of the voltmeter. It readily follows from (24), remembering that $\mathbf{N}$ is independent of θ, that

$$\mathrm{F} = \rho\mathrm{N},$$

(27) $$\text{and} \quad \frac{1}{\mathrm{F}}\frac{d\mathrm{F}}{d\theta} = \frac{1}{\rho}\frac{d\rho}{d\theta},$$

so that the general expression (13) for the steady torque is the same as

(28) $$\mathrm{T}_s = \frac{1}{\rho}\frac{d\rho}{d\theta}\overline{\mathbf{CF}} = \frac{d\rho}{d\theta}\overline{\mathbf{CN}},$$

which in the case of the above voltmeter (with ρ negligible compared with unity) becomes

$$\mathrm{T}_s = \pm\frac{d\rho}{d\theta}\mathrm{KV}^2. \tag{29}$$

Next assume the moving coil circuit to be metallically closed through the secondary of a transformer the coils of which have a constant mutual inductance M. Suppose the primary coil of this transformer to be traversed by a current $\mathbf{A}$. Let R be the resistance of the moving coil circuit, but for the present let us assume the self-inductance of this circuit negligible. We then have

$$\mathrm{R}\mathbf{C} = -\dot{\mathbf{F}} + \mathbf{E}, \quad \text{where} \quad \mathbf{E} = \mathrm{M}\dot{\mathbf{A}}.$$

Also by (25)

$$\dot{\mathbf{F}} = \rho\mathbf{V},$$

and by (13) and (28)

$$\mathrm{T}_s = \frac{1}{\rho}\frac{d\rho}{d\theta}\overline{\mathbf{CF}}.$$

But

$$\begin{aligned}\mathrm{R}\overline{\mathbf{CF}} &= -\overline{\mathbf{F}\dot{\mathbf{F}}} + \overline{\mathbf{EF}} = \mathrm{M}\overline{\dot{\mathbf{A}}\mathbf{F}} \\ &= -\mathrm{M}\overline{\mathbf{A}\dot{\mathbf{F}}} = -\mathrm{M}\rho\overline{\mathbf{AV}},\end{aligned}$$

so that

$$\mathrm{T}_s = \pm\frac{d\rho}{d\theta}\frac{\mathrm{M}}{\mathrm{R}}\overline{\mathbf{AV}}, \tag{30}$$

or T_s is a measure of the power in watts associated with the current $\mathbf{A}$ and the voltage $\mathbf{V}$. The ambiguity of sign merely implies that the secondary of the transformer can be connected up in two ways.

Certain assumptions have been made in establishing the formulæ (29) and (30), and it remains to show what influence any error in these assumptions has on the action of the instrument.

I have already discussed the theory and construction of iron cored shunt magnet instruments, and described a rather long experimental investigation of their behaviour, in two papers published by the Institution of Electrical Engineers (vols. 34 and 36). In one of these papers the theory of the voltmeter, and of its error, were very fully dealt with, and it will suffice to add here that these voltmeters are most satisfactory instruments. They need but a negligibly small current to work them, and are very sensitive, especially when constructed so as to have a weak control, and for use with an optical pointer.

The theory of the wattmeter merits much fuller consideration. The properties of magnetic fields due to shunt excited electromagnets have a wider interest than that arising from the use of these magnets for a particular purpose, such as that of a voltmeter or a wattmeter. If a magnetic field can be caused and controlled by an applied voltage in accord-

ance with a strict mathematical law [by which is meant a law which in essence involves no factors dependent upon the magnetic or other variable physical properties of the medium, and which, therefore, is independent of changes in these physical properties], such a field must be found useful, sooner or later, for a variety of measuring purposes. No more searching test can well be applied to such a field than an investigation of its action in connection with a wattmeter, for no other electrical instrument is required to work under such severe conditions. Its indications must be correct for all values of no fewer than five variables, viz., amperes, volts, power factor, frequency, and wave form. Moreover, while the deflecting forces corresponding with the power to be measured diminish with the power factor of the load, this is not the case with those corresponding with the error of the instrument. The true power for a given current and voltage is proportional to $\cos\phi$, while the most important part of the error is proportional to $\sin\phi$, so that this actually increases as $\cos\phi$ diminishes, and becomes relatively very important for low power factors.

Now to determine T_s we have, besides (28), the equations

(31) $$\mathbf{V} = r\mathbf{A}_m + \dot{\mathbf{N}} \qquad \mathbf{F} = \rho\mathbf{N},$$
$$\mathrm{R}\mathbf{C} + \mathrm{L}\dot{\mathbf{C}} = -\dot{\mathbf{F}} + \mathbf{E}.$$

E is the electromotive force in the secondary of the transformer the primary of which is traversed by the main current $\mathbf{A}$. For an air core transformer $\mathbf{E}$ will be strictly proportional to $\dot{\mathbf{A}}$. If the magnetic circuit of the transformer contains iron, the flux $\mathbf{N}_a$ enclosed by the secondary coil will not be strictly proportional to $\mathbf{A}$, but we can always put (see (2))

(32) $$\mathbf{N}_a = \mathrm{M}\mathbf{A} + \mathbf{n},$$

where $\mathbf{n}$ is in quadrature with $\mathbf{A}$ and

$$\mathbf{E} = \dot{\mathbf{N}}_a = \mathrm{M}\dot{\mathbf{A}} + \dot{\mathbf{n}}.$$

From the above equations (31) we can omit $\dot{\mathbf{F}}$, since, as already shown (19 and 15), we can calculate the part of T_s due to $\dot{\mathbf{F}}$ separately. Its value will be

$$T_1 = -\frac{1}{\rho}\frac{d\rho}{d\theta}\mathrm{L}C_1^2,$$

where C_1 is calculated from

$$\mathrm{R}\mathbf{C}_1 + \mathrm{L}\dot{\mathbf{C}}_1 = -\dot{\mathbf{F}},$$

or since $\mathbf{C}_1$ and $\dot{\mathbf{C}}_1$ are in quadrature and the ratio of the magnitudes of $\mathrm{L}\dot{\mathbf{C}}_1$ and $\mathrm{R}\mathbf{C}_1$ is very small,

$$\mathrm{R}^2C_1^2 = \dot{\mathrm{F}}^2 = \rho^2\mathrm{V}^2$$

with sufficient accuracy. Hence the torque due to $\dot{\mathbf{F}}$ is

(33) $$T_1 = -\rho\frac{d\rho}{d\theta}\mathrm{L}\frac{\mathrm{V}^2}{\mathrm{R}^2},$$

a quantity determined by V alone. It is not dependent on A or on the power factor of the load, nor is it dependent upon the frequency or wave form of V, assuming rA_m is negligible. If we omit $\dot{\mathbf{F}}$ from equations (31), these become

(34)
$$\mathbf{V} = r\mathbf{A}_m + \dot{\mathbf{N}}, \qquad \mathbf{F} = \rho\mathbf{N},$$
$$\mathrm{R}\mathbf{C} + \mathrm{L}\dot{\mathbf{C}} = \mathrm{M}\dot{\mathbf{A}} + \mathbf{n} = \dot{\mathbf{N}}_a.$$

The above may be regarded as equations between vectors, and remembering (7) and the quadrature relationship of vectors like $\mathbf{C}$ and $\dot{\mathbf{C}}$, the complete figure can be easily indicated as in fig. 3, where ϕ is the phase difference of the current $\mathbf{A}$ and the main voltage $\mathbf{V}$, and the small phase errors of the instrument, which for the sake of clearness are greatly exaggerated in the figure, are denoted as follows:—

(35) ϕ_m, the angle between $\dot{\mathbf{N}}$ and $\mathbf{V}$ due to the resistance of the magnet coil.

ϕ_s „ „ $\dot{\mathbf{N}}_a$ and $\mathbf{C}$ „ self-inductance of the moving coil circuit.

ϕ_i „ „ $\mathbf{N}_a$ and $\mathbf{A}$ „ hysteresis of the transformer core.

These angles must each be made very small if the wattmeter is to be even approximately correct, so that we shall regard them as small quantities of the first order, and neglect their squares and products compared with unity. Hence it follows, with the aid of (34), that to this degree of approximation

$$\mathrm{RC} = \mathrm{M}\dot{\mathrm{A}} = \mathrm{M}p\mathrm{A}, \qquad \mathrm{V} = \dot{\mathrm{N}} = \mathrm{N}p,$$
$$\mathrm{F} = \rho\mathrm{N} \quad \text{or} \quad \mathrm{F}p = \rho\mathrm{V},$$

and that these magnitudes are the same as if ϕ_m, ϕ_s, ϕ_i, were all zero.

Now the torque depends upon $\overline{\mathbf{FC}}$, that is upon FC $\cos(\phi+\phi_\epsilon)$, where $(\phi+\phi_\epsilon)$ is the phase difference between $\mathbf{F}$ and $\mathbf{C}$. The magnitudes F and C are the same as if the phase errors were zero. It will be seen from fig. 3, assuming as a first approximation that all the vectors lie in one plane, that $(\phi+\phi_\epsilon)$ is the complement of the angle separating C and $\dot{\mathrm{N}}$, since $\dot{\mathrm{N}}$ is in quadrature with N and F. Hence

$$\phi+\phi_\epsilon = \phi+\phi_m+\phi_s+\phi_i,$$

and, since all the phase errors are small,

$$\cos(\phi+\phi_\epsilon) = \cos\phi\,(1-\phi_\epsilon\tan\phi),$$

(36)
$$\text{where} \quad \phi_\epsilon = \phi_m+\phi_s+\phi_i,$$

Hence, by (28), (31), and (33), the complete expression for T_s is

(37)
$$\mathrm{T}_s = \pm\frac{d\rho}{d\theta}\frac{\mathrm{M}}{\mathrm{R}}\mathrm{W}(1-\phi_\epsilon\tan\phi)-\rho\frac{d\rho}{d\theta}\mathrm{L}\frac{\mathrm{V}^2}{\mathrm{R}^2},$$

where ϕ_ϵ is given by the preceding equation and where $W = VA \cos \phi$, the true power in watts.

In establishing the above, we have assumed that all the vectors of fig. 3 lie in one plane. This, in general, will not be the case, nor, indeed, as a rule, will a three-dimensioned figure suffice. But if all the angles forming ϕ_ϵ are small quantities of the first order, it will be seen that the length of any projected vector will bear to the length of the corresponding unprojected

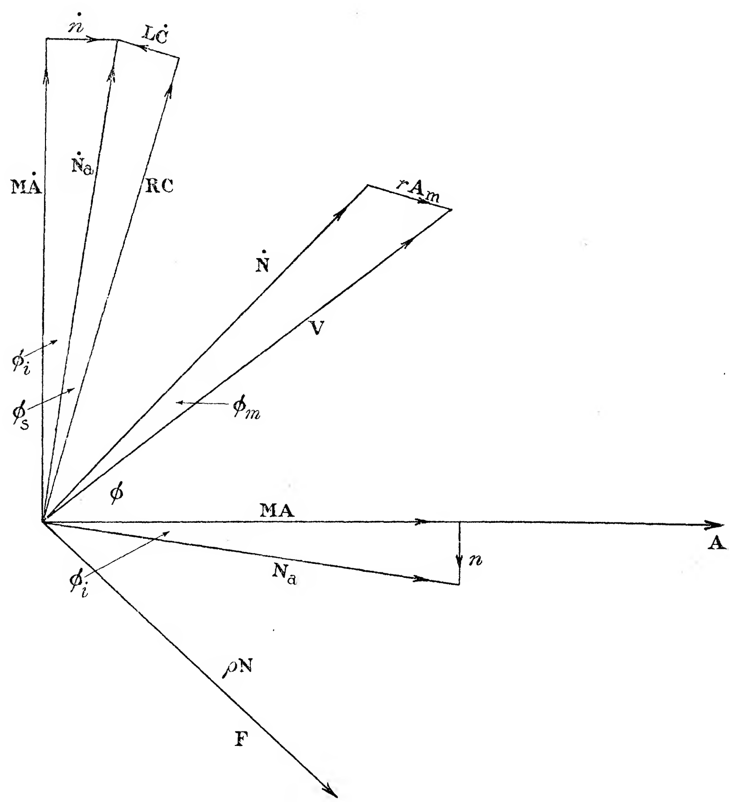

Fig. 3.

vector a ratio $1-\epsilon^2$, where ϵ^2 is a small quantity of the second order. Moreover, this will also be true of the projected values of the phase errors ϕ_m, ϕ_s, ϕ_i. Neglecting such quantities, the formula (37) is still accurate, as also (36), the equation for ϕ_ϵ, provided the latter is regarded as a vector equation, so that ϕ_ϵ can never exceed, and must, in general, be less than, the numerical sum of the separate phase errors. The full analytical investigation quite bears out these statements, but as its working involves much detail, and does not bring out any new point, it will be sufficient to

indicate a shorter proof in which advantage is taken of the smallness of the fractions denoting the phase errors.

The value of T_s must evidently be a function of V, A, ϕ, and the phase errors, and, hence, if we can neglect squares and products of the latter, we can, by Taylor's theorem, expand T_s and put

$$T_s = {}_0T_s[1+\lambda\phi_m+\mu\phi_s+\nu\phi_i],$$

where ${}_0T_s$ is the value of T_s, assuming all the phase errors to be zero, and where λ, μ, ν do not involve these quantities. It only remains to determine the values of these coefficients, and it is clear that we can find any one of them by assuming two of the quantities r, L, n equal to zero, and solving the equations (34) so modified. It is easy to show that each coefficient can be expressed by the formula

$$-(1-\epsilon^2)\tan\phi,$$

where ϵ^2 is a positive quantity, which may be zero, which is always very small, and which, in fact, can nearly always be neglected.

Thus, to find λ, the coefficient of ϕ_m, we put L and $\mathbf{n}$ each equal to zero in (34), and have the equations

$$\mathbf{V} = r\mathbf{A}+\dot{\mathbf{N}}; \quad \mathrm{R}\mathbf{C} = \mathrm{M}\dot{\mathbf{A}}; \quad \text{and} \quad \mathbf{F} = \rho\mathbf{N}.$$

Thus
$$\overline{\mathbf{FC}} = \rho\frac{\mathrm{M}}{\mathrm{R}}\overline{\mathbf{N}\dot{\mathbf{A}}} = -\rho\frac{\mathrm{M}}{\mathrm{R}}\overline{\dot{\mathbf{N}}\mathbf{A}}$$

$$= -\rho\frac{\mathrm{M}}{\mathrm{R}}\dot{\mathrm{N}}\mathrm{A}\cos(\phi+\phi_m'),$$

where $\phi+\phi_m'$ is the angle between the vectors $\mathbf{A}$ and $\dot{\mathbf{N}}$. Referring to fig. 3, it will be seen that though the vectors $\dot{\mathbf{N}}$, $\mathbf{V}$, and $\mathbf{A}$ may not lie in one plane, they can be properly represented in a three-dimensioned figure, and that $\phi+\phi_m'$ cannot exceed $\phi+\phi_m$, and must, in general, be less, or the ratio of ϕ_m' to ϕ_m must be less than unity. Also the magnitudes V and $\dot{\mathrm{N}}$ must be equal, since ϕ_m is small and A_m is approximately perpendicular to V. In other words, $\dot{\mathrm{N}}$ is independent of ϕ_m, and it readily follows that

$$T_s = {}_0T_s\frac{\cos(\phi+\phi_m')}{\cos\phi} = {}_0T_s[1-\phi_m'\tan\phi],$$

so that
$$\lambda = -\frac{\phi_m'\tan\phi}{\phi_m} = -(1-\epsilon^2)\tan\phi.$$

By a similar process the coefficients μ and ν may be obtained, and their values will be found to satisfy a similar formula. The result is to completely establish (36) and (37) with the limitation already stated in regard to the former equation; that it must be regarded as a vector

equation, so that ϕ_ϵ can never be greater, and must, in general, be less than the numerical sum of ϕ_m, ϕ_s, and ϕ_i.

Part IV.—Experimental Verification.

A moving coil electromagnet instrument, having the characteristics already described, may be constructed of thin sheet iron stampings, as illustrated in fig. 5, in which shaded areas denote the windings in section. The magnetic circuit contains only one air gap. To reduce the reluctance of this as much as possible, its section is increased by extending the poles, so that one almost surrounds the other as shown. The polar distance across the gap is reduced to the smallest value consistent with the free motion of the moving coil. This coil is rectangular in shape, and turns about one

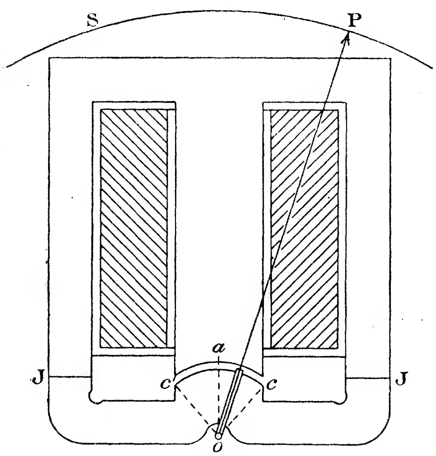

Fig. 4.

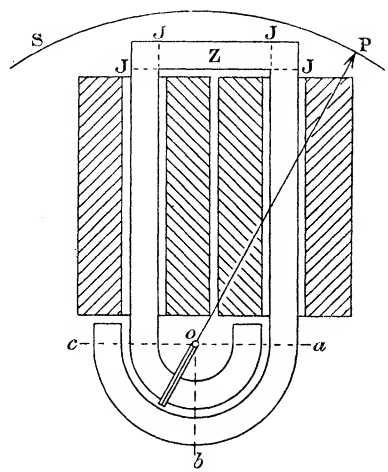

Fig. 5.

of its long sides as an axis (shown at O), the opposite side moving in the arc-shaped air gap between the poles. A pointer OP is attached to the coil and reads on a suitable scale PS. The magnetic joints, indicated at J, needed for constructional purposes, are made as good as possible by building up successive stampings, so as to overlap at the joints. The resistance of the magnetising coil (consisting of two windings, one on each limb) is reduced by using as much metal in the coil as the available winding space allows. Such constructional features ensure two desirable results. The ratio of the resistance of the coil to its impedance is made small, and the ratio of the magnetic leakage to the total magnetic flux is diminished as much as possible. The latter ratio cannot be assumed to be quite constant for different magnetising currents, so that, unless it is a small fraction, the

flux density at a particular part of the air gap may not always be sufficiently proportional to the total flux enclosed by the coil. The instruments actually made have been constructed to several designs, the earliest of which is indicated in fig. 4. In none of these instruments has any adverse effect due to variation of magnetic leakage been observed. But the instrument illustrated in fig. 4 is more likely to show such an effect than that illustrated in fig. 5. The magnet of fig. 4 consists of two blocks of stampings with butt joints at JJ, and measurement shows that while most of the magnetic flux through the centre limb crosses the narrow circular air gap in which the moving coil turns, an appreciable portion of the total flux leaks to the side limbs by longer air paths. In the magnet of fig. 5 this leakage must bear a much smaller ratio to the total flux.

The Inductance of the Moving Coil.

If the field coil of such an electromagnet be excited by an alternating voltage, and the moving coil be closed through a small resistance, this coil will be found to turn (in accordance with (16)) till the flux it encloses is as small as possible. Thus, in each of the cases represented in figs. 4 and 5, the coil will turn towards the position Oa. By increasing the resistance of the moving coil circuit the turning force can be easily made negligible, since, very approximately, the torque is inversely proportional to the square of this resistance (see (15)).

If the moving coil is connected with the terminals of a condenser and the field magnet coil be excited, the moving coil will be found to turn so as to enclose the greatest possible flux (see (18)). For the case represented by fig. 5 the coil will turn to the position Oc, while for that represented by fig. 4 the coil will turn to the nearer of the two positions Oc, Oc'.

These effects are quite in accordance with the theory previously given, but do not indicate any special facts about the self-inductance of the moving coil. More information is obtainable on this point by testing the formula (11)

$$\mathbf{T} = \mathbf{C}^2 \frac{dL}{d\theta},$$

where **C** is the moving coil current and L is the self-inductance of the coil. In order to test this formula, it is necessary to eliminate the torque, due to the interaction of the moving coil current, and the flux due to the current in the magnetising coil. This can be done by using currents of two different frequencies for the moving and field coils. Under these circumstances the only torque acting on the moving coil is that given by the above formula,

and by observing the movement of the coil we can thus find in which direction L increases.

Such tests were made upon an instrument constructed like fig. 5. The turning moment was found to be so minute that the controlling springs had to be removed, the current being passed into the moving coil by means of fine wires exerting no appreciable control. The moving system had also to be carefully balanced. The moving coil contained 20 turns, and the maximum current that could be safely passed through it was about 0·2 ampere. The magnetic force due to this current was feeble (the length of the magnetic circuit was about 15 inches, and the magneto-motive force was spent mostly on the gap), so that the permeability of the iron was low. The first test was made with the field magnet unexcited and with the field coil open circuited. An alternate current of 0·2 ampere through the coil caused it to turn to a position between O*b* and O*c*, about 30° from O*b* as actually shown in fig. 5. The position of maximum self-inductance would, theoretically, be along O*c* if the only magnetic circuit comprised the long iron path from pole to pole and the single air gap between the poles. The self-inductance is, however, partly due to the local magnetic circuit crossing the air gap twice, once on each side of the moving coil, and this portion of the self-inductance will be greatest when the coil lies along O*b*. Hence, under the actual conditions, the position of maximum self-inductance must lie between O*b* and O*c*, as was indicated by the test. The field magnet coil (having 2000 turns and a resistance of 8·5 ohms) was then electrically closed, either by a short circuit, or by a small resistance, this latter consisting, in one case, of a single accumulator producing through the field coil a current of about a quarter of an ampere, and, in another case, of the low resistance armature of an excited alternator applied direct to the field coil. An alternate current of 0·2 ampere (produced by a second alternator) when passed through the moving coil caused the latter in each of these cases to turn to the position O*b*. That is to say, whenever the field magnet coil was electrically closed (the total resistance of the circuit being in all cases small) the position of maximum self-induction of the moving coil was invariably O*b*, whether the electromotive force impressed on the field magnet coil circuit was steady, alternating, or zero. Thus the self-inductance of the moving coil was dependent upon the resistance of the field coil circuit. To explain this, it must be remembered that a closed coil of low resistance has, approximately, the shielding effect of a perfectly conducting sheet. If we assume the resistance of the field coil to be zero, and the coil to be short circuited (or what is mathematically equivalent if a voltage be applied to it, the value of which is independent of current), it is theoretically impossible for the number of lines of force enclosed by the coil to be in

any way altered, unless there is an unbalanced electromotive force in the circuit, in which case the rate of change of the flux must be such as to cause a reactive electromotive force exactly balancing that which is impressed. In such case the closure of the field coil circuit would perfectly control the flux through the core, and this flux would be uninfluenced by the currents in a neighbouring circuit such as that of the moving coil.

The theory of the instrument already given shows that it is necessary to carefully consider the inductance of the moving coil circuit. It is found possible to construct the instruments so that this inductance has no appreciable influence on the deflections. Such a result may, at first, seem unlikely. The inductance, directly or indirectly, affects the deflection in three ways.

In the first place, the reactance causes a shift of phase of the moving coil current, and a resultant error similar to that which arises in a wattmeter of the ordinary dynamometer type.

Next there is the torque given by the formula

$$\mathrm{T}_s = \mathrm{C}^2 \frac{d\mathrm{L}}{d\theta},$$

where C is the moving coil current, and θ is the deflection.

Finally, there is the torque due to inductive action given by (15) and (33),

$$\mathrm{T}_s = -\frac{1}{\mathrm{F}} \frac{d\mathrm{F}}{d\theta} \mathrm{LC}^2 = -\rho \frac{d\rho}{d\theta} \mathrm{L} \frac{\mathrm{V}^2}{\mathrm{R}^2}$$

and $$\mathbf{F} = \rho \mathbf{N} \quad \text{or} \quad \mathrm{F}p = \rho \mathrm{V},$$

where $\mathbf{F}$ is the electromagnet flux, traversing the moving coil, the value of which increases with the number of turns, and, therefore, with the self-inductance of this moving coil.

Moreover, it is necessary to consider whether the moving coil can even be said to have a true, *i.e.*, constant, self-inductance, for the coil is always in the close neighbourhood of iron masses whose induction density, and, therefore, permeability, are continually varying throughout the period.

It must be borne in mind that the self-inductance of a coil is merely a coefficient, convenient when constant, for connecting the magnetising current with the magnetic flux resulting from it, and that what is necessarily associated with a magnetising current is not a magnetic flux but a magnetising force. The current causes no flux if the circumstances are such that the magnetising force due to the current merely calls into existence, by reaction, an equal and opposite magnetising force. The case is analogous to that of the weight of a body which causes no acceleration in it when at rest on a table. If there is no extra flux due to the current in the coil, this coil must be

regarded as devoid of self-inductance. Such a result is, in the main, true of the coils of a potential transformer. The variation of the secondary current does not alter the flux through the core, since this flux is essentially controlled by the primary voltage; all that happens is a change in the primary current of such a character and amount as to exactly neutralise the change in the magnetising force due to the alteration of the secondary current. The flux through the core, as shown diagrammatically in fig. 6, consists of a

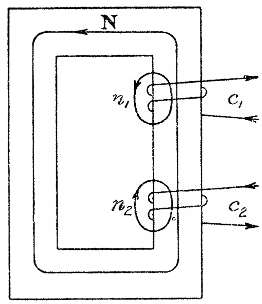

FIG. 6.

main portion **N** threading both the primary and secondary coils C_1 and C_2 and traversing the best magnetic circuit, and also of two local, or leakage fluxes $\mathbf{n}_1$ and $\mathbf{n}_2$, each of which threads the windings of one coil only, and each of which traverses a bad magnetic circuit the reluctance of which is almost entirely due to that of an air path and, therefore, essentially constant. The flux **N** is determined by the primary voltage, and is independent of the currents in the coils C_1 and C_2, but the flux $\mathbf{n}_1$ is strictly proportional to the current in C_1, while the flux $\mathbf{n}_2$ is similarly proportional to the current in C_2. If one of the coils, say C_2, is on open circuit, the self-inductance of the other corresponds with a small flux $\mathbf{n}_1$ strictly proportional to the current, together with a large flux **N** which, owing to the variable permeability of the iron, is not proportional to the current. Under these conditions the self-inductance is large but not constant. If, however, one of the coils, say C_1, is excited by an alternating voltage, the self-inductance of the coil C_2 is solely due to the small flux $\mathbf{n}_2$. This leakage flux is proportional to the secondary current, so that energy is not dissipated when the flux alternates, and the coil has a constant self-inductance in spite of the presence of iron masses. But the inductance is very small, and it is possible that under some circumstances it may be actually less than would be the case if the iron masses were removed. The iron, with its voltage controlled flux **N**, diminishes the flux $\mathbf{n}_2$ resulting from the current in the secondary coil, since it occupies paths otherwise available for this flux. For instance, if the primary coil C_1 is wound as in

fig. 5, so as to well cover the core, and the secondary consists of a few turns round the primary coil, the reluctance of the leakage path $\mathbf{n}_2$ will, in such case, be so great that the secondary coil will be essentially devoid of self-inductance.

By those accustomed to transformers, and similar apparatus, it is recognised* that the self-inductance of the coil arises from the leakage flux, and not from the main flux; but I know of only one set of experimental tests in which the matter has been thoroughly investigated. These tests were carried out at King's College, London, in 1892, and are described in a report written by the late Dr. John Hopkinson for the Westinghouse Company.

The description in question is much more than an ordinary commercial report. It is a record of a searching experimental investigation on two transformers. Among the points examined was the instantaneous relationship of the leakage fluxes and the coil currents. Unfortunately this investigation has only been published in the technical press,† so that it may be useful to state briefly the experimental result of interest here.

Referring to fig. 6, if $\mathbf{V}_1$ and $\mathbf{V}_2$ are the voltages, $\mathbf{A}_1$ and $\mathbf{A}_2$ the currents, and r_1 and r_2 the resistances of the primary and the secondary coils respectively, we have

$$\mathbf{V}_1 = r_1\mathbf{A}_1 + \dot{\mathbf{N}} + \dot{\mathbf{n}}_1,$$

with a similar equation for $\mathbf{V}_2$. The experiments showed that the leakage flux $\mathbf{n}_1$ was proportional to the current $\mathbf{A}_1$ for every instant of the period and for different magnitudes of the current $\mathbf{A}_1$. This was proved by determining the values and wave-forms of the different quantities required by means of Joubert's instantaneous contact method, using a quadrant electrometer as indicator. The curves for $\mathbf{V}_1$ and $r_1\mathbf{A}_1$ were separately determined in the usual manner. The curve for $\dot{\mathbf{N}}$ was obtained with the aid of an open circuit coil surrounding the flux $\mathbf{N}$, or by an equivalent method, and the curve for $\dot{\mathbf{n}}_1$ was obtained from the others by using the above equation. The curve for $\mathbf{A}_1$ was then differentiated to get the curve $\dot{\mathbf{A}}_1$, and the latter was finally compared with the curve for $\dot{\mathbf{n}}_1$. The two curves were found to coincide, and the comparison made was the more convincing because of the irregular and complicated shape of the curves. The tests were carried out

* As, for instance, in the much used "short-circuit" methods of testing alternate current machinery. The first of these short-circuit methods was pointed out by myself in 1892 in reference to the efficiency testing of transformers. See 'Proc. Inst. Elec. Engineers,' vol. 21, p. 741.

† See 'Electrician,' vol. 29, June 24 and July 1, 1892. I understand that most of the tests were actually carried out by Professor E. Wilson, who has since published in the 'Electrician' for February 15 and February 22, 1895, an account of some additional experiments which constitute further verification of some of the points here referred to.

not only with the secondary coil open circuited, but also with different load currents taken from this coil. The value of $\mathbf{n}_1$ was found proportional to $\mathbf{A}_1$, and that of $\mathbf{n}_2$ was found proportional to $\mathbf{A}_2$, at each instant, and under all circumstances, assuming constant voltage excitation of one of the coils.

It follows that the self-inductance of the moving coil is due to the part of the flux circulating through it, but not traversing the core surrounded by the field coil. In the case of fig. 5 the self-inductance will be greatest when the coil lies along O*b*, and in that of fig. 4 the corresponding position will be O*a*, but the self-inductance will be larger in the latter case owing to the leakage flux which passes through the moving coil, and traverses the closed iron magnetic circuit formed by the outer limbs of the magnet, without traversing the central core whose flux is controlled by the field coil circuit.

In several cases the impedance (and hence reactance) of the field coil was tested experimentally and found to compare satisfactorily with that calculated from the dimensions of the gap, the reluctance of the iron part of the magnetic circuit being negligible for the induction densities used in the tests. It was hence possible to deduce the reactance Lp of a field coil having the same number of turns as the moving coil. A quarter of this value will be the working reactance of the moving coil in the position O*b* of fig. 5, since the two halves of the air gap will be in parallel for one magnetic circuit and in series for the other. It was thus possible to calculate the approximate value of the inductance of the moving coil for different values of θ, and to show that the values of L, and of $dL/d\theta$, were too small to affect the accuracy of the instruments actually constructed.

In some cases the value of Lp was measured directly. This was not a simple matter. It was necessary to use an alternate current method. It was useless to attempt to compare the impedance of the coil with its resistance, for these two quantities only differed by about one part in 5000, since the reactance was only about 2 per cent. of the resistance. A bridge method was actually used, the coil to be tested forming one arm, the other three consisting of non-inductive resistances, two of which measured about 1000 ohms each. All the resistances were suitable for, and were used with, currents of about 0·1 ampere. A pressure of 100 volts, alternating at a frequency of 50, was applied to the bridge. The cross conductor was arranged to have a resistance of about 200 ohms to reduce shunting errors, and contained an instrument capable of measuring pressures as low as 0·0002 volt applied to this conductor. So far as I am aware, no instrument for alternating currents is obtainable which is at all suitable for such a measurement. The voltage was measured after rectification by a suitable

commutator,* with the aid of a delicate galvanometer. To measure the minimum cross voltage it was necessary to make an extremely fine adjustment of the bridge. This was obtained by means of a sliding contact on a potentiometer wire 2 metres long and of about 9 ohms resistance. The minimum cross voltage, divided by the current through the coil under test, gave the reactance Lp of this coil.

An instrument of the type indicated in fig. 5 was examined by the above method. The moving coil contained 20 turns and had a resistance (including leads) of 11 ohms. It was fixed in the position Ob and its reactance was tested when the field coil was (i) on open circuit, and (ii) on short circuit. The values measured were respectively 0·160 and 0·136 ohm for a frequency of 50 cycles per second. The difference between these numbers is the part of the reactance due to the main magnetic circuit of the magnet, and its small value is attributable to the low permeability of the iron under the feeble magnetic forces due to the moving coil current. A coil of 15 turns, wound round the yoke of the magnet (at Z, in fig. 5), was also tested with currents of the same frequency. The resistance of this coil was 0·4 ohm, and its reactance was found to be 0·033 ohm with the field coil open circuited, and 0·014 ohm with the field coil short circuited. The two coils were afterwards put in series (opposing), and the reactance of the combination was found to be 0·147 ohm for the field coil short circuited, and 0·150 ohm for the field coil open circuited. The coils so connected were essentially non-inductive as regards their mutual magnetic circuit, so that the two above values should be equal to each other, and also to the sum of the numbers 0·136 and 0·014 found for the coils separately. This auxiliary coil was originally wound round the magnet for another purpose, being intended for use in series with the moving coil, and to be so connected that the voltages induced by the magnet in these coils opposed each other and balanced for a certain position of the moving coil. In this way the small error (see (15) and (33)) due to the voltage induced in the moving coil circuit could be eliminated or made negligible. The device was, however, found to be needless, owing to the insignificant amount of the error in question. A fixed auxiliary coil wound round the inner pole (near the line Ob in fig. 5) might in special cases prove useful, since essentially it would serve to remove from the circuit of the moving coil not only the voltage induced by the electromagnet, but also the self-inductance of the moving coil.

* For the commutator and voltmeter methods here referred to, see "The Measurement of Small Differences of Phase," 'Phil. Mag.,' January, 1905. They have been much used in connection with the present investigation for the determination of minute alternating voltages.

Measurement of Deflecting Torque.

The residual magnetism properties of an electromagnet such as is indicated in fig. 4 or fig. 5 are most striking, but the distribution of the magnetism in the air gap must be the same, whatever current may be passing through the field coil. All the instruments were calibrated by direct current methods. A steady current was passed through the field coil and maintained constant, while observations were made of the deflections in degrees due to various measured steady currents through the moving coil. The scale marking was made proportional to the current producing the corresponding deflection. The scale so obtained has been found in all cases accurate for alternate current use, being directly proportional for wattmeters, and proportional to the square of the volts for voltmeters. The air gap is bounded by two sets of stampings, those of each set having circular edges of the same radius, so that with good construction the air gap can be made very approximately of the same radial distance everywhere, and the wattmeter scale almost exactly equally divided. Tests have proved this to be the case.

For the purpose of verifying the formulæ given above for the torque T_s acting on the moving coil, tests were also made of the voltage induced in this coil by the alternating electromagnet. For this purpose the alternating voltage on the magnet coil was kept constant, and the voltage of the moving coil measured for different values of the deflection θ. The ratio of the latter voltage to the former is ρ. The relation between ρ and θ was found in all cases to be very approximately linear. In the case of an instrument like fig. 5, ρ increased as θ altered from Oa to Oc, while, for an instrument like fig. 4, ρ passed through zero for some deflection denoted by Oa. The relation between ρ and θ was so nearly linear that it was found sufficiently accurate for the purpose of an approximate verification of the formulæ for T_s to take as the value of $d\rho/d\theta$ the alteration in ρ for a change in θ of one radian, or $57^\circ{\cdot}3$. The formulæ given are in absolute units, so that the unit torque is one erg per radian. It is convenient to measure the electrical quantities in commercial units, and the torque in "gramme centimetres," so that the formulæ for T_s must be multiplied by $10^7/g$, or approximately by 10,200.

One instrument tested had a field coil of 990 turns. The moving coil contained 20 turns, and its working range was 140°. This coil was fixed at deflections of 0°, 20°, 40°, 60°, 80°, 100°, 120°, and 130°, and the measured values of ρ expressed as percentages were respectively 0·38, 0·50, 0·64, 0·77, 0·91, 1·05, 1·18, and 1·23. These values will be found to plot very well on a straight line, the slope of which corresponds with a rate of 0·39 per cent. per radian.

Using σ for shortness to denote 10,200 $d\rho/d\theta$, the above tests imply that σ is 39·8. Formulæ (28), (29), and (30) for T_s expressed in gramme-centimetres become respectively

$$\text{(i) } \sigma\overline{\mathbf{CN}}\text{; (ii) } \sigma \mathrm{KV}^2\text{; (iii) } \sigma\frac{\mathrm{M}}{\mathrm{R}}\mathrm{W}.$$

The strength of the spring was first tested mechanically by placing a wire rider, weighing about 20 milligrammes, on the pointer at a measured distance from the axis, and carefully adjusting the instrument till the pointer and the axis were each horizontal, first with and then without, the rider. In this way the torque for 80° deflection was measured as 0·126 gr. cm., corresponding with 0·142 gr. cm. for 90°. The instrument was then tested as a voltmeter with a standard condenser of 0·5 microfarad for the moving coil circuit. A number of tests made for deflections near 80° yielded as a mean result that 79 volts produced a deflection of 81°. Substituting in the formula $\sigma = 39{\cdot}8$; $K = 0{\cdot}5 \times 10^{-6}$; $V = 79$, we get 0·124 gr. cm. for 81°, or 0·138 gr. cm. for 90°. A test on the instrument as a wattmeter on a non-inductive load, measuring the current and voltage with hot wire instruments, resulted in a deflection of 126° for 94 volts and 18 amperes, or for 1692 watts, when the transformer was such that M was approximately 0·305 millihenry, and R, the total resistance of the moving coil circuit, was 108·8 ohm. These numbers, when substituted in the above formula, give 0·189 gr. cm. for 126°, or 0·135 gr. cm. for 90°. Thus the mechanical tests and those using formulæ (ii) and (iii) yield values respectively 0·142, 0·138, and 0·135 gr. cm. for a deflection of 90°.

This instrument was afterwards altered. A moving coil of 44 turns replaced the original one of 20 turns and stronger controlling springs were used, the torque for 90° measuring approximately 1·10 gr. cm. The value of σ was not again measured, but it may be assumed to increase in proportion to the number of turns on the moving coil, and hence may be taken as 39·8 × 44/20, or 87·6. Two transformers were successively used with this instrument, the resistance of the moving coil circuit being adjusted in each case till the instrument gave a deflection of exactly 50° for 1000 watts. The values of M for the two transformers were measured approximately as 0·765 and 1·00 millihenry. The corresponding resistances for the moving coil circuit were 114·1 and 150·6 ohms. On substituting in formula (iii) above the two values of the torque for 50° work out to be 0·587 and 0·582 gr. cm., or, for 90°, 1·056 and 1·047 gr. cm. When used as a voltmeter with a condenser of 1·43 mf., the instrument gave a deflection of 95° for a pressure of 97 volts. On substituting in formula (ii) these numbers will be found to correspond with a torque of 1·12 gr. cm. for 90°.

A test was also made using formula (i) in connection with an instrument referred to in subsequent tests (see Table II). It was found that when 107 volts at a frequency of 52 were applied to the fixed coil and moving coil in series, the resultant deflection was 58°. The value of σ found for this instrument was approximately 56, and the torque required to deflect the spring 90° was 1·24 gr. cm. The field coil was such as to take 90 milliamperes for a voltage of 200 when the frequency was 50 cycles per second, hence for 107 volts and 52 cycles per second the value of C in (i) is 0·0463 ampere. The value of N is such that $2\pi \times 52\text{N} = 107$, or N is 0·328. Hence, σCN is 0·848 gr. cm. for 58°, or 1·32 gr. cm. for 90°. This result would have been nearer 1·24, the value found from the mechanical test, if allowance had been made for the phase difference ϕ_i between the magnetising current C, and the resultant flux N. The angle ϕ_i, judging from measurements made on other instruments, is about 13°, and though large enough to be fatal for a wattmeter having an electromagnet of the ordinary series, or current controlled type, is not such as to make $\cos\phi_i$ differ more than 3 or 4 per cent. from unity.

No attempt was made to prove exact coincidence between the mechanically and electrically measured values of the torque. The tests in question really constituted absolute measurements, and in order to carry them out properly it would have been necessary to most carefully check all the instruments, and subsidiary measurements, made use of. This was not done, since only an approximate verification was aimed at. Many such tests were made on the various instruments constructed. The results above given are taken from the more concordant tests. The indications of the commercial instruments used were accepted without verification.

Condenser Compensation.

The most serious error arising in the action of the wattmeter on circuits of low frequency and low power factor is that due to the resistance of the field coil. The self-inductance of the moving coil circuit tends to increase the effect of this error, not to compensate it. The voltage $\mathbf{V}$ applied to the field coil of resistance r produces a current $\mathbf{A}_m$ which magnetises the core and causes a reactive voltage $\mathbf{U}$ or $\dot{\mathbf{N}}$. These quantities are related by the equation

$$\mathbf{V} = r\mathbf{A}_m + \mathbf{U},$$

and in fig. 7 are represented respectively by the vectors BP, BO, and OP. If the number of turns on the field coil is m the product $m\mathbf{A}_m$ can be represented to some other scale by BA drawn along BO, and is the number of ampere turns round the core needed to magnetise it to the extent represented by the

reactive voltage $\mathbf{U}$. The error of the instrument arises from the phase difference between $\mathbf{U}$ and $\mathbf{V}$, not from the fact that the magnitude of $\mathbf{U}$ is less than that of $\mathbf{V}$. The ratio between U and V is independent of the current and the power factor, and is essentially the same for all voltages. In all cases it differs inappreciably from unity, and any difference that exists is taken account of in the standardisation of the instrument.

For a given frequency the relationship between the total magnetising ampere turns BA round the core, and the reactive core voltage OP is quite fixed by the magnetic circuit, even if there are several coils round the core traversed by independent currents. Suppose that in addition to the primary

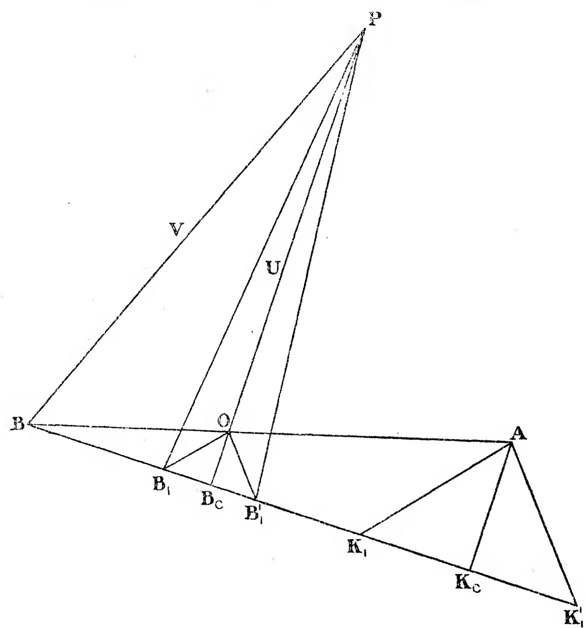

Fig. 7.

exciting coil already referred to there is a second winding of m_2 turns round the core, and that the terminals of this coil are joined to a condenser. The voltage $\mathbf{V}_1$ on the primary necessary to produce a reactive voltage $\mathbf{U}$ due to the core, will now be given by

$$\mathbf{V}_1 = r\mathbf{A}_1 + \mathbf{U},$$

where $\mathbf{A}_1$ is the new current in the primary. Since $m\mathbf{A}_m$ is the total number of ampere turns necessary to produce the core voltage $\mathbf{U}$, we have

$$m\mathbf{A}_1 + m_2\mathbf{A}_2 = m\mathbf{A}_m,$$

where $\mathbf{A}_2$ is the condenser current. The current $\mathbf{A}_2$ will necessarily be in quadrature with the core voltage $\mathbf{U}$, since it in phase with $\dot{\mathbf{U}}$, and hence

$m_2\mathbf{A}_2$ will be represented by a vector BK_1, perpendicular to OP, while its magnitude will be proportional to the capacity of the condenser applied to the secondary winding. Since the vectors BA and BK_1 represent respectively $m\mathbf{A}_m$ and $m_2\mathbf{A}_2$, it follows that K_1A will represent $m\mathbf{A}_1$ and that if OB_1 be drawn parallel to AK_1 to meet BK_1 in B_1 this vector will denote $r\mathbf{A}_1$, while the vector B_1P will represent $\mathbf{V}_1$. By increasing the capacity of the condenser attached to the secondary winding, the point B_1 may be made to reach the line OP at B_c and even to cross it to B_1', so that the phase difference between $\mathbf{V}_1$ and $\mathbf{U}$ can be reduced to zero and can even be reversed in sign until from an angle of "lag" it becomes an angle of "lead." It is thus possible by the use of a condenser of the right capacity to compensate not only the phase error of the electromagnet, but also the extra phase error due to the inductance of the moving coil circuit. The adjustment will hold for different voltages, but will only be exact for one frequency. This is all that is needed in most cases, since one of the most constant properties of an alternate current system of distribution is its frequency.

In the case considered the length of the line OP is from 100 to 200 times the length of OB, so that to a close approximation the lines BP, B_1P, OP, etc., are all of equal length. Also the angle BOP does not much exceed a right angle, and hence BA and BK_c are essentially equal. The angle OBB_c is about 13°, while the angle BPO is only about a quarter of a degree. If the field coil of the electromagnet be excited at constant voltage and frequency, the reactive voltage U can thus be regarded as fixed, whatever the capacity K of the condenser applied to the secondary winding. For a zero value of K the primary current will have a certain value represented by BA. For a particular value K_c of the capacity, the primary current will be a minimum represented by AK_c, the line BK_c representing to some other scale the capacity K_c. For any other capacity K_1, represented by BK_1, the corresponding primary current will be represented by AK_1. Thus, if the magnet be excited at constant voltage and frequency, and measurements of the primary current be taken for various values of K, the curve obtained by plotting the primary current or ampere turns as ordinates, and the corresponding values of K as abscissæ, will be catenary-shaped with a minimum ordinate for the capacity K_c needed to compensate the phase error of the magnet. Such curves are well known in connection with alternate current measurements. When suitable scales are chosen the curve must also be such that the ordinate is equal to AK_1 when the abscissa is equal to BK_1 in fig. 7. The value of K_c obtained will be inversely proportional to the square of the current frequency used, since the ampere turns BA needed to produce a given reactive voltage U will vary inversely as the frequency, while the ampere

turns BK_1 due to a capacity K_1 will be directly proportional to the frequency. For a similar reason the capacity needed will be inversely proportional to the square of the number of turns used on the secondary winding. All these points have been fully verified by actual tests.

The results given below in Table I and fig. 9 are taken from a set of tests made on an electromagnet provided with a moving coil, and connected up as a voltmeter of the magnetostatic type previously described. The connections of the coils are shown diagrammatically in fig. 8. The field coil

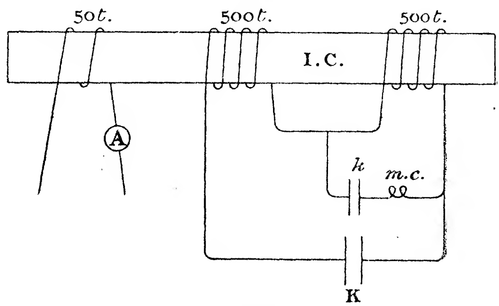

Fig. 8.

consisted of two coils of 500 turns each, the two coils being twin wound round the core so as to ensure that each coil enclosed the same flux. These coils were permanently connected in series. The moving coil, *m.c.*, was connected to the terminals of one of the windings of 500 turns through a condenser k of 0·5 microfarad capacity. The core of the magnet was wound with an extra coil of 50 turns. The pointer was found to deflect to a certain mark on the scale when 80 volts were applied to 1000 turns (or 40 volts to 500 turns). The deflection for a given voltage was the same for *any* ordinary frequency of the voltage used for the test. This deflection was utilised in the tests to adjust the reactive voltage of the core to a constant value. Experiments were then made by passing an alternating current of A amperes through the 50-turn coil. The current frequency was kept constant at 80 cycles per second, and A was adjusted by means of suitable resistances until the pointer was steady at the mark on the scale, the current being read by a hot wire ammeter. The values found for A for different capacities of K microfarads applied to the field coil of 1000 turns are given in the first two columns of Table I. A third column gives the corresponding values of 50A or the ampere turns associated with the 50-turn coil. The value of K was known with fair accuracy, and was due to various parallel combinations of condensers which had been tested by ballistic methods some months previously. The half microfarad condenser used with the

500-turn coil for the moving coil circuit caused a small magnetising current round the field windings. The values given for K represent the capacities actually connected with the 1000-turn coil, and these should be each increased by 0·5/4 or 0·125 microfarad to get numbers proportional to the ampere turns due to the condenser currents.

Table I.

A. amperes.	K. mf.	50A. amp. turns.	A. amperes.	K. mf.	50A. amp. turns.
3·1	0·0	155·0	0·70	4·33	35·0
2·16	1·58	108·0	0·85	5·02	42·5
2·2	1·62	110·0	1·0	5·35	50·0
1·83	2·0	91·5	1·70	6·64	85·0
1·69	2·33	84·5	1·87	6·93	93·5
1·28	3·02	64·0	2·79	8·55	139·5
0·95	3·58	47·5	3·23	8·97	161·5
0·93	3·62	46·5			

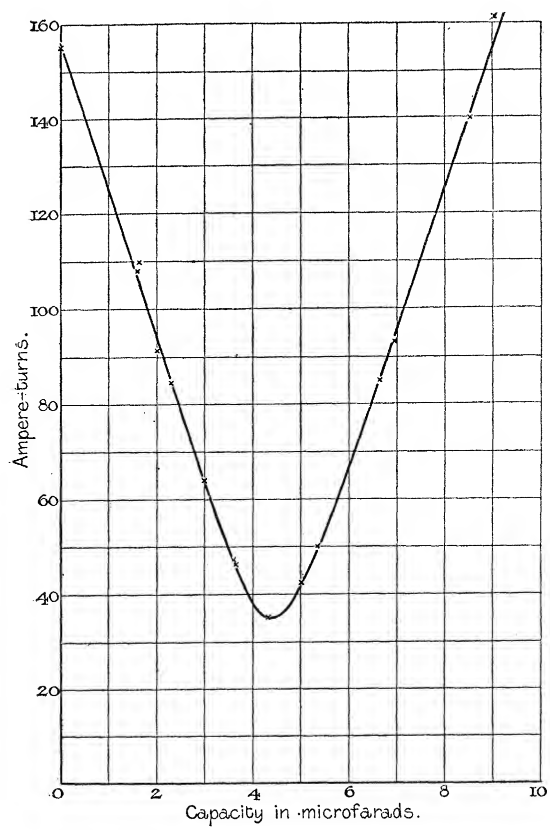

FIG. 9.

The results are shown plotted in fig. 9. When the current A has its minimum value, reference to fig. 7 will show that A is in phase with the reactive voltage. Hence the power wasted in the core may be obtained by multiplying the minimum number of ampere turns, 35, by 0·080 the voltage per turn round the core. This power works out to be 2·8 watts for 80-cycle circuits for the induction density corresponding with this frequency and 0·080 volt per turn. The loss thus obtained agrees closely with other measurements which have been made upon the core losses of the electro-magnet in question.

A more convenient method of testing one of these instruments for the capacity K_c needed for compensation is (see fig. 10) simply to put the field coil *f.c.* in series with the moving coil *m.c.* and to apply various condensers K to the terminals of the field coil only, when a constant voltage V is applied to the two coils in series, the frequency being kept constant. The impedance of the moving coil is quite negligible compared with that of the field coil. Hence for constant voltage and frequency the reactive voltage **U** (fig. 7) is constant. The magnetic field, which is in quadrature with **U**,

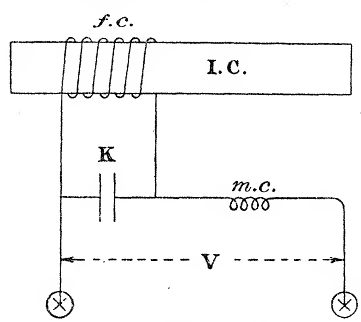

Fig. 10.

is represented by a vector drawn along the line BK and is fixed both in phase and magnitude. The vector BA represents the field coil current, which in this case includes the condenser current, but the moving coil current is represented by AK, for a capacity K, applied to the field coil. Since the magnet field is represented by a vector of constant length drawn along BK, the torque acting on the moving coil will be represented by the projection of AK_1 on BK_c or by K_1K_c. It will thus reverse in sign as the capacity increases through K_c, the value needed for compensation, and, moreover, there will be a linear connection between the capacity used and the moving coil torque as indicated by the deflection on the calibrated wattmeter scale.

The results given in Table II, and shown plotted in fig. 11, illustrate one of many tests made on a number of instruments by this method.

Table II.

K	0	0·50	1·00	1·43	1·96	2·46	2·96
θ	58	36·5	13·5	−1·0	−23·8	−48·0	−68·0

The voltage was kept constant at 107 volts and the frequency was always adjusted approximately to 52 cycles per second, by means of a resonance frequency meter. K is the capacity in microfarads shunted to the field magnet coil, and θ is the deflection of the instrument. The negative

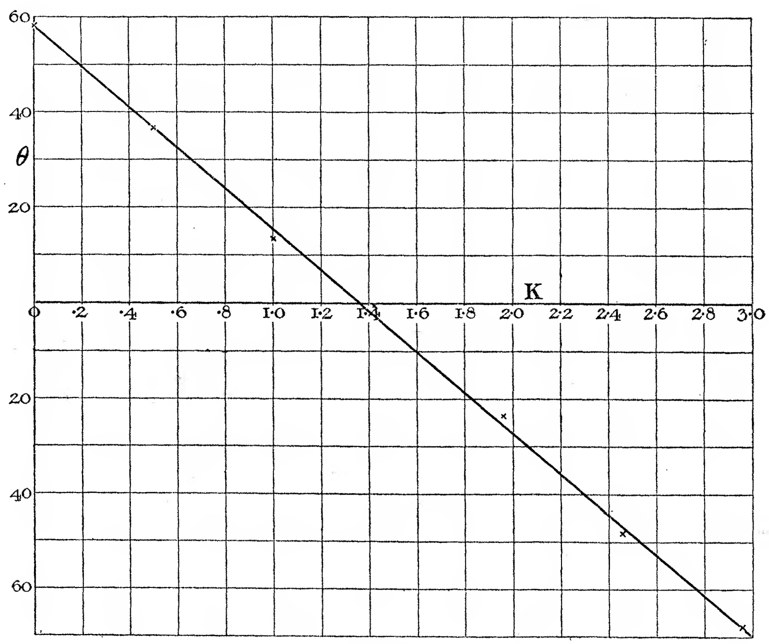

Fig. 11.

readings were obtained by reversing the connections of the moving coil. The instrument was the same as that referred to later in the tests on compensated wattmeters, the field winding consisting of four coils in parallel, each coil containing 2000 turns. From the plotted results it appears that K_c is 1·37 microfarads for a frequency of 52, corresponding with $1{\cdot}37 \times (52)^2/(50)^2$, 1·48 microfarads for circuits of frequency 50. The ampere turns due to the condenser winding if the frequency is 50, the voltage is 200, and $K = 1{\cdot}48$ mf. will be $2000 \times 1{\cdot}48 \times 10^{-6} \times 314 \times 200 = 186$. This is approximately the value of mA_m for this frequency and voltage, since tests showed that for 200 volts at 50 cycles per second the value of A_1 was

90 milliamperes, corresponding with 180 ampere turns. The agreement is satisfactory in view of the fact that the frequency was not adjusted with any special care.

Compensated Wattmeters.

The foregoing theory shows that a wattmeter of the type here considered, if correctly calibrated on non-inductive loads, will, when used on inductive loads, of power factor $\cos\phi$, give a reading denoting

$$W[1-\phi_\epsilon \tan\phi],$$

where W is the true power in watts, and ϕ_ϵ is the phase error of the instrument given by (36). The wattmeter reads low for lagging currents, and high for leading currents in direct contrast to the behaviour of a wattmeter of the ordinary dynamometer type.

When an additional coil is wound round the core and connected to the terminals of a condenser, the value of ϕ_ϵ is given by the formula

$$\phi_\epsilon = \phi_m+\phi_s+\phi_i-\phi_c,$$

where ϕ_c is the phase angle corresponding with the condenser winding and which is proportional to the capacity of the condenser attached to this winding. The ratio of ϕ_c to ϕ_m will approximately be that of the ampere turns due to the condenser to the ampere turns needed to magnetise the core.

By using a suitable condenser it is possible to reduce ϕ_ϵ to zero and even to reverse its sign, that is to make the wattmeter read high for lagging currents and low for leading currents, like an ordinary dynamometer wattmeter. This has been experimentally verified on several wattmeters the electromagnets of which differed greatly in shape. The following results were obtained on a wattmeter having an electromagnet shaped like that represented in fig. 5. This instrument had been wound for use as a multirange voltmeter. The magnet was wound with four coils each of 2000 turns. Two of these coils were wound on each limb, one coil being completely wound on the bobbin before the winding of the other was commenced. The resistances of the four coils are 74·5, 93·0, 71·5, and 90·4 ohms, the two lower resistances corresponding with the inner coils, and the others with the outer coils. With all four coils in parallel it was found that 200 volts produced a current of 0·090 ampere when the current frequency was 50, so that 180 ampere turns were needed to produce a flux density in the core corresponding with 10 turns per volt at 50 cycles per second, and it follows that the impedance of the winding is 2222 ohms at this frequency. The parallel resistance of all four coils is 20·3 ohms. The two inner coils when in parallel measure 36·5 ohms and the two outer coils

in parallel measure 45·9 ohms. Hence the value of ϕ_m, or the ratio of resistance to impedance, is as given in Table III for different combinations of coils and for frequencies of 50 and 80.

Table III.—Values of ϕ_m.

Frequency.	50.	80.
	Per cent.	Per cent.
All coils in parallel...........	0·92	0·57
Inner ,, ,, ...	1·65	1·03
Outer ,, ,, ...	2·07	1·29

The above values are higher than those obtained in other instruments of the same type, the reason being that the winding consisted of a large number of turns of thin wire insulated for high potential working, so that the space occupied by the insulation was unusually large compared with the space occupied by the copper.

The moving coil was connected up in series with the secondary of a current transformer, and a non-inductive resistance, the total resistance of the circuit being 118 ohms. The transformer had a primary and secondary coil each of 94 turns, the former being suitable for current up to 10 or 15 amperes, and the latter consisting of fine wire. The reactance Lp of the secondary was 0·24 ohm at 50 cycles per second. The mutual inductance of the coils was 0·765 millihenry. The value of ϕ_s due to the reactance of the secondary was 0·24/118, or 0·203 per cent. at 50 frequency, corresponding with 0·325 per cent. at 80 frequency. The transformer actually used contained iron (and an air gap) in its magnetic circuit, so that the value of ϕ_i must be allowed for. Measurements* showed that its value varied from 0·3 to 0·4 per cent. in accordance with the currents used.

If for the moment the self-inductance of the moving coil itself be neglected it will be seen that for a frequency of 80, and using all four coils

* Owing to the length of the present paper, no reference is made to the conditions for accuracy of a quadrature transformer containing iron in its core, but the tests made in this connection have been most numerous. To measure ϕ_i an air core transformer was chosen, the coils of which had about the same mutual inductance as those of the iron core transformer to be tested; an alternating current was passed through the two primaries in series, and measurements were made of the phase difference of the two secondary voltages. This was done by means of the voltmeter methods already referred to, taking due care to allow for the phase differences introduced by the self-inductance of the secondary coils. The testing networks used were essentially either Felici or Christie balances, and in some cases the phase difference measured was increased, and in others diminished, by the self-inductance of the secondaries.

in parallel for the field coil, the values of ϕ_m, ϕ_s, and ϕ_i are respectively 0·57, 0·32, and 0·35 per cent. The sum of these numbers is 1·24 per cent. If the field coil consists only of the two outer coils in parallel, this number, owing to the increase of ϕ_m, becomes 1·96 per cent. Tests of the uncompensated wattmeter on circuits of frequency 80 and for loads of power factor $\cos\phi$, have shown percentage errors given by the formula

$$\phi_\epsilon \tan\phi$$

for values of ϕ_ϵ in close agreement with the above numbers.

An approximate value of the reactance of the moving coil can be obtained from the impedance of the 2000 turns of the field winding. This, for a frequency of 50, is 2222 ohms. The moving coil consisted of 40 turns. If the field coil contained only 40 turns, the impedance would be $2222 \times (40)^2/(2000)^2$, or 0·89 ohm. The moving coil, when in the central part of the gap, will have its reactance a maximum, and approximately equal to a quarter of the above value. It will thus be 0·22 ohm for a frequency of 50, or 0·35 ohm for a frequency of 80. Since the resistance of the moving-coil circuit is 118 ohms, the phase error due to the coil's reactance will be 0·3 per cent. for 80 cycle circuits, and this will be its maximum value. As already stated, the total phase error ϕ_ϵ is the vector sum, and is less than the numerical sum, of the different phase errors. I have generally found that the value of ϕ_ϵ, experimentally deduced from tests of the wattmeter on inductive loads, agrees well with that calculated from adding the separate phase errors, when that due to the reactance of the moving coil is neglected.

In order to test the iron-cored instrument (I.C.W.) as a compensated wattmeter, the outer coils were put in parallel, and used as the field winding, to which the voltage was applied, while the inner coils, also in parallel, were applied to the terminals of a condenser. The tests were made, using as a standard wattmeter a Mather Duddell instrument (M.D.W.), constructed by Messrs. Paul. In order that the tests on lagging current loads could be immediately succeeded by tests on leading current loads, and *vice versâ*, the circuits were arranged as shown in fig. 12.

Two similar single-phase alternators, rigidly coupled so as to run together, generated two voltages $\mathbf{V}_1$ and $\mathbf{V}_2$ of the same frequency and approximately in quadrature. The magnitudes of $\mathbf{V}_1$ and $\mathbf{V}_2$ could be varied independently by adjusting the exciting currents of the two machines. The current $\mathbf{A}$ was produced by the voltage $\mathbf{V}_1$, and passed through (i) the primary P of the current transformer of the I.C.W.; (ii) the current coils C of the M.D.W.; (iii) a large non-inductive resistance R consisting of glow lamps; and (iv) a small adjustable carbon resistance CR. This current was approximately in

phase with the voltage $\mathbf{V}_1$, but there was a slight angle of lag owing to the inductance of the coils in the circuit. The voltage $\mathbf{V}$ for the pressure coils of the two wattmeters was obtained from the mains M, N, which were connected up to the two alternators with the aid of a reversing switch, R.S., so arranged that, for one position of the switch, $\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$, and for the other position, $\mathbf{V} = \mathbf{V}_1 - \mathbf{V}_2$. The vector figure is indicated in fig. 13. The current $\mathbf{A}$ was in each case approximately 45° out of phase with the voltage $\mathbf{V}$, leading for one position of the switch ($\mathbf{V} = \mathbf{V}_1 - \mathbf{V}_2$) and lagging for the other position ($\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$). Its magnitude in the various tests varied from 7 to 9 amperes, being in all cases adjusted by the carbon resistance CR till the M.D.W. was balanced. The voltage V for one position of the switch was about 230 volts, and for the other about 200 volts. The frequency was in all cases adjusted to 80 cycles per second. The pressure circuit of the

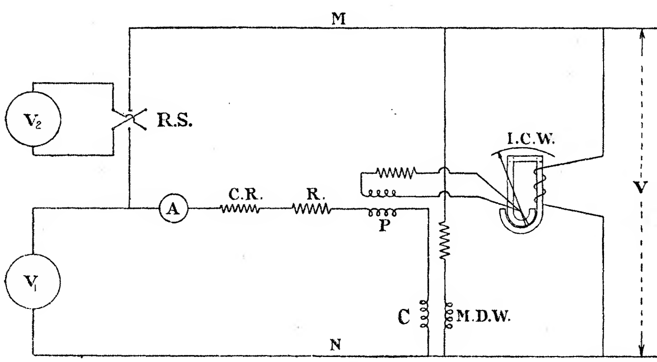

Fig. 12.

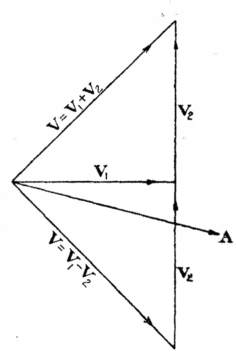

Fig. 13.

M.D.W. contained a non-inductive resistance of 9000 ohms, which was amply sufficient to make this circuit essentially non-inductive for a frequency of 80. One terminal of this resistance was connected to the main M, so that the pressure coil of the M.D.W. had one of its ends joined directly to the main N, and was thus at essentially the same potential as the current coil C. This precaution was found necessary to make negligible any electrostatic forces acting on the moving system of the M.D.W., this moving system being large and delicately suspended. The constant of the M.D.W. had been carefully determined on a non-inductive load, using the same hot wire instruments to measure the pressure and current as were afterwards employed to measure the same quantities for the inductive tests. The power factor $\cos\phi$ of the load was determined in each case from the readings of the volts, amperes, and of the watts, as indicated by the hot wire instruments and the M.D.W. From the power factor $\cos\phi$ the value of $\tan\phi$ was deduced, this value

being considered positive for lagging currents ($\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$) and negative for leading ones ($\mathbf{V} = \mathbf{V}_1 - \mathbf{V}_2$).

In Table IV, below, the first column shows the capacity in microfarads joined up to the two parallel connected inner coils wound round the electromagnet core. The reduced readings of the M.D.W. for a deflection of exactly 40° on the I.C.W. are next given, arranged in three columns, in order to separate the results obtained with the three kinds of load corresponding with the voltage used. The last two columns show the value of the power factor as deduced from the readings of voltmeter, ammeter, and wattmeter, and the corresponding value of tan ϕ.

Table IV.

Capacity K.	Readings of M.D.W.			cos ϕ.	tan ϕ.
	$V = V_1 + V_2$.	$V = V_1$.	$V = V_1 - V_2$.		
0	—	—	63 ·0	0 ·852	−0 ·613
	—	63 ·4	—	1 ·000	0
	64 ·6	—	—	0 ·755	0 ·869
	—	—	62 ·9	0 ·796	−0 ·758
	65 ·0	—	—	0 ·799	0 ·753
	—	—	62 ·4	0 ·796	−0 ·758
	64 ·6	—	—	0 ·799	0 ·753
2 ·0	62 ·0	—	—	0 ·748	0 ·884
	—	63 ·4	—	0 ·994	0 ·100
	62 ·5	—	—	0 ·755	0 ·815
	—	—	64 ·8	0 ·823	−0 ·688
	62 ·7	—	—	0 ·748	0 ·884
	—	—	64 ·8	0 ·823	−0 ·688
0 ·9	63 ·6	—	—	0 ·748	0 ·884
	—	63 ·6	—	1 ·000	0 ·000
	—	—	63 ·6	0 ·822	−0 ·688

The results show that (i) the uncompensated instrument (K = 0) is more sensitive for leading currents than for lagging ones; (ii) when a capacity of 2 microfarads is used the reverse is the case; and that (iii) when a capacity of 0·9 microfarad is used, the instrument is perfectly compensated for circuits of frequency 80. It was easy to test the effect on the reading due to suddenly switching on the condenser K, and there was no doubt that for lagging currents switching on the condenser caused an increase of the deflection of the I.C.W., while for leading currents the reverse effect took place. When the load was essentially non-inductive no effect due to switching on a condenser could be observed. This is only natural, since

when $\phi = 0$ the error $\phi_\epsilon \tan \phi$ vanishes, whatever the value of ϕ_ϵ. But though the nature of the effect due to the condenser current could thus be easily demonstrated, it needed a set of tests such as those indicated in Table IV to actually measure its amount. The numbers given are in each case the means of several observations, and only a few of the tests taken are given. This more especially applies to the observation with K = 0·9 microfarad. These were repeated many times, with the result that it was impossible to say under the actual conditions of test whether the I.C.W. was more, or was less, sensitive with load currents lagging than with load currents leading. It now only remains to show that the differences actually observed are in accordance with the error formula.

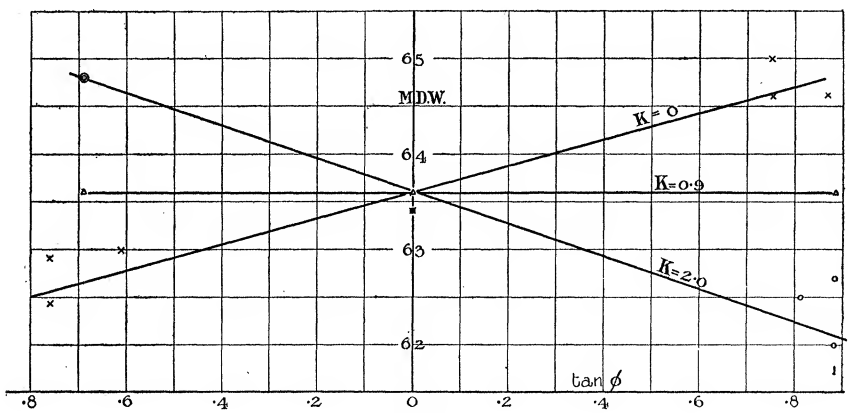

Fig. 14.

The results given in Table IV are plotted in fig. 14, abscissæ representing values of tan ϕ and ordinates the corresponding wattmeter readings for the standard deflection of the I.C.W. These results are found to plot fairly closely on three straight lines, each having the ordinate 63·6 for the zero value of tan ϕ. The resultant phase errors calculated from the slopes of the lines in fig. 15 will be found to be 2·05, 2·68, and 0 per cent. for values of K respectively, equal to 0, 2, and 0·9 microfarads. Owing to the nature of the error formula $\phi_\epsilon \tan \phi$, the mean value of ϕ_ϵ can be best obtained numerically by taking the mean wattmeter reading and the corresponding mean value of tan ϕ for each set of tests, and combining the results as indicated in Table V.

R denotes the ratio of the difference between the wattmeter readings to the corresponding difference between the tangents, while the ratio of R to 63·6 is ϕ_ϵ the phase error.

Now, as already shown, the calculated values of ϕ_m, ϕ_s, and ϕ_i add up to

Table V.

K.	Wattmeter readings.		$\tan\phi$.		Differences in—		R (ratio).	ϕ_ϵ (per cent.).
	Lag.	Lead.	Lag. +	Lead. −	Reading.	$\tan\phi$.		
0	64·73	62·77	0·784	0·710	1·96	1·494	+1·31	+2·06
2·0	62·40	64·80	0·750	0·688	−2·40	1·438	−1·67	−2·62
0·9	63·60	63·60	0·884	0·688	0	1·572	0	0

1·96 per cent., disregarding the self-inductance of the moving coil, which will increase the number by an amount less than 0·3 per cent. This figure agrees closely with the phase error of 2·06 per cent. as actually measured for $K = 0$. The value of ϕ_c, or the shift in the value of ϕ_e, caused by applying condensers of 0·9 and 2 microfarads to the secondary winding, will be seen from the tests to be 2·06 per cent. and 4·68 per cent. respectively. These numbers can readily be compared with the values calculable from the capacities. The value of ϕ_m alone, as shown by Table III, is 1·29 per cent. for an exciting winding consisting of the two outer coils in parallel and for the frequency of 80. Also, it was found that 180 ampere turns round the field winding produced a magnetic flux corresponding with 0·1 volt per turn at a frequency of 50, and, therefore, with 0·16 volt per turn at a frequency of 80. The alternation of such a flux would induce in 2000 turns a pressure of 320 volts at the latter frequency. Assuming the wave form of the voltage approximately sinuous, as was the case in the actual tests, a condenser of 1 microfarad applied to the 2000 turns under the conditions assumed would take a current of $2\pi \times 80 \times 10^{-6} \times 320$ ampere, or 0·161 ampere, corresponding with 332 ampere turns for the coil of 2000 turns. The value of ϕ_c for a condenser of 1 microfarad is, therefore, very approximately given by

$$\frac{322}{180}\,1{\cdot}29 \text{ per cent.} = 2{\cdot}32 \text{ per cent.}$$

Hence, condensers of 0·9 and 2 microfarads would, at the frequency of 80 assumed, produce values of ϕ_c, respectively equal to 2·09 per cent. and 4·64 per cent., which are essentially the same as the values actually found. The testing conditions were such that the closeness of the agreement is of no special significance, but it will be apparent that within the limits of experimental error the action of the wattmeter corresponds exactly with the formula given for it.

Conclusion.

The foregoing tests were all made with alternate currents of the low frequencies usual in commerce, on instruments of the non-reflecting type, each having a moving system subject to the control of a strong spring. Few tests have yet been made on reflecting instruments in which a weak control is used for the moving coil. Under such conditions much greater sensitiveness can naturally be secured, but, owing to causes already mentioned, the sensitiveness reached will not even then be comparable with that obtainable with the best direct current galvanometers. To measure very minute alternate currents or voltages it must ultimately prove necessary to use instruments of the heterostatic type. For such purposes the shunt-excited electromagnet seems specially suited, and additional interest in consequence attaches to the foregoing examination of the behaviour of voltage-controlled magnetic fields.

The precision of direct current measurements is mainly due to the use of null methods. Analogous methods have been suggested, or can easily be devised, for alternate current testing. But such methods are not used. The advantage of a null method arises from the possibility of fully utilising the sensitiveness of an instrument for the purpose of measuring or indicating a small difference between two nearly equal quantities. It follows that when, as with alternate currents, instruments sufficiently sensitive to indicate such a difference do not exist, null methods are really of little value. In fact, when the testing voltmeter is such that the deflection depends on the square of the voltage, a direct deflection method is more sensitive than a balance method, assuming the voltage tested is not large enough to over-deflect the instrument. By making use of the properties of separately excited voltage-controlled fields it seems possible to construct voltmeters for alternate current working which are quite as sensitive as the corresponding direct current instruments, and which are also such that the deflecting forces are proportional to the first power of the voltage tested. The special difficulties due to phase can be readily overcome, since it is easy to apply in succession to the field coil two voltages of known relative magnitude and phase. For the purpose of null methods the magnitudes of these voltages need only be very roughly determined.

The instruments yet made have been suitable only for low-frequency circuits. For high-frequency working special difficulties will arise, while others will disappear. For low frequencies the chief difficulty is to make negligible the phase error ϕ_m, represented approximately by the ratio of resistance to impedance. For high frequencies ϕ_m will become quite

negligible, but on the other hand ϕ_s, the phase error due to the inductance of the moving coil circuit, will become serious, as also possibly will eddy current effects in the core of the electromagnet. Eddy currents in the core only affect the accuracy of the magnet in so far as they affect the value of ϕ_m, provided the core is so well laminated that the distribution of the magnetic field is the same for direct as for alternate currents. With very high frequency currents the value of ϕ_m will be negligible, and, possibly, in some cases, there will be no need to use iron in the core, so that eddy currents will not occur. The increase of the inductance phase error ϕ_s for high frequencies is more serious, but on the other hand it is possible to compensate it, not only for *one* but for *all* frequencies, by means of the condenser winding. It can readily be shown that while ϕ_m varies inversely as the frequency, the ratio of ϕ_c to ϕ_m varies directly as the square of the frequency, so that ϕ_c is directly proportional to the frequency like ϕ_s. Hence, it should prove possible by using a suitable condenser in conjunction with a special winding to neutralise the phase error of the instrument for all frequencies for which ϕ_m is negligible.